# DESCRIPTIVE GEOMETRY

SEVENTH EDITION

H. W. MILLER

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# DESCRIPTIVE GEOMETRY

# DESCRIPTIVE GEOMETRY

BY

## H. W. MILLER, M.E.

*Author of "Mechanical Drafting," "Erosion of Guns," "Railway Artillery," "Seacoast Artillery," "Mobile Artillery," "The Paris Gun," etc.*

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# PREFACE

The first edition of this textbook was prepared in 1910. At that time numerous teachers of engineering drawing were convinced that modernizations of the books on descriptive geometry written by Professors Warren and Church in 1860 and 1864 were required. Specialization of engineering for the various divisions of industrialization was sufficiently pronounced to indicate the need to modernize the older textbooks on many subjects. Few teachers of the time, however, could look forward thirty years and appreciate entirely the form and the outline which textbooks in mathematics, physics, and chemistry should take. As a consequence the modernizations of 1910 have required further modification as developments have pointed the way.

The present revision has been prepared again with the idea of adapting the subject to the needs of the engineer of today. The author has departed from the consistently abstract method of treatment of Professor Church, whose book he was admittedly revising in 1910; but has not felt inclined to adopt what appears to him to be too high a degree of specialization in some of the textbooks of today, which employ so-called direct methods. If the phrase "direct method" means anything, it must be a "modern adaptation to a specific problem," which is the role of a formula or a handbook. Descriptive geometry is taught to all engineers in approximately the same fashion. It is a part of the curriculum in the engineering colleges throughout the world. Few engineering students can possibly know what variety of engineering work they will follow. Very few follow the variety which they expect to follow. Too high a degree of specialization may therefore be more dangerous than useful. For these reasons the writer has retained enough of Professor Church's plan of academic presentation for the purpose of the teacher, who must employ conventional classroom methods and models, but at the same time has dealt with each problem, both in discussions and illustrations, in such fashion as to enable the engineer to use the

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book as a reference in the solution of any of the geometrical problems of design, no matter what his specialization may be.

Students of the subject of descriptive geometry are urged to accept the subject as one of the most utilitarian of their curriculum. It is not merely a disciplinary subject as some individuals insist. It is the one subject of the approximately forty making up an engineering curriculum which enables the engineer to solve the multitude of variations of the five geometrical problems essential to the design of any engineering project. Unless those geometrical problems can be solved by someone, nothing can be designed. The subject is therefore not academic in the least, although its applications may not be fully appreciated by students. Furthermore, it is not a difficult subject, and students should not rate it as such. The percentage of failures in this subject in college is actually less than in most mathematical subjects. Students may therefore approach the subject as a necessity in their curriculum, as highly utilitarian, and not difficult.

The author wishes to acknowledge his indebtedness to the members of the staff of the Department of Engineering Drawing at the University of Michigan for their fine, thorough, and effective assistance in the preparation of this revision.

HENRY W. MILLER

Particular attention is invited to the twenty-three sheets of test exercises given as Part IV of this book. They have been designed to supplement the discussions and illustrations of the text proper and to give the student an opportunity to test out his understanding of theory and to acquire skill in using it. It is urged that students and instructors give these test exercises a thorough trial.

H. W. M.

ANN ARBOR, MICHIGAN
*December, 1940*

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# DESCRIPTIVE GEOMETRY

## INTRODUCTION

A modern engineering drawing comprises a group of related plane-geometry drawings and is made for one or more of the several purposes of study, design, construction, or maintenance of an engineering project. The individual drawings are inadequate because they are but partial descriptions of the object or project. The entire group of related drawings, however, constitutes a complete and entirely adequate description for the purpose for which the drawings were made. It is to the point to emphasize that the individual drawings are plane-geometry drawings, that individually they are inadequate, but that collectively they are entirely adequate to describe the project involved and to answer the purpose for which they were intended.

One of the earliest evidences of an appreciation of the advantages of associating several plane views of the same structure is to be found in the book on architecture by Marcus Vitruvius, who lived in the time of the Roman Emperor Augustus and was appointed by him as superintendent of military engines for the Roman armies. Vitruvius was a most painstaking engineer and builder, and his voluminous book on architecture deals in great detail with practically everything even remotely related to the planning and construction of public buildings. There can be no doubt that he made and used plan and elevation drawings of his buildings, but there is nothing to indicate that he appreciated the possibility of solving geometrical problems by the principle of associating like-scale plan and elevation drawings in a mathematical fashion. It is unfortunate that the historians who have told us so much about social conditions of the ancient empires have chosen to tell us so little about current practices in engineering. The qualities of the Babylonian, Egyptian, Greek, and Roman structures which have endured until now indicate high degrees of engineering talent and well-developed and widely recognized principles of design.

There must have been further developments in the mathematical principles of modern engineering drawing in the fifteen centuries between the times of Vitruvius and of Albrecht Dürer, the famous German artist, but there seems to be no record of them. In two of Dürer's books, "Geometry and Perspective," published in Nuremberg in 1525, and "Human Proportions and How to Draw the Human Figure," published also in Nuremberg in 1528, there is evidence that this writer did appreciate to a limited extent the possibilities of associating plan and elevation drawings. There appears in the latter publication of Dürer a three-view, first quadrant drawing of the significant features of a human head.

If the known extent of appreciation of the possibilities of the principle of associating several like-scale views of a structure seems small for fifteen centuries, it should be remembered that there was but little if any need for such a principle. Even in the time of Dürer the tempo and the quality of engineering and of science were such as to make but little demand upon the ingenuity of human beings. It seems incredible that the zero of mathematics was introduced into Europe by way of Italy as late as the sixteenth century, after a long and laborious wait of some twelve centuries from its inception in India. Perhaps this slow development is the indication that mathematics as a whole would not be accommodated to the increasing tempo of modern demands on engineering until the period of widespread discovery and the wholesale occupation of new lands all over the world (1400–1800) had increased the demands upon ingenious people for labor- and time-saving ideas.

Whatever the reason for the tardy discovery of the essential principle of modern engineering drawing, it probably must be accredited to a brilliant young mathematician, Gaspard Monge, who was born at Beaune in France in 1746. He was educated at the colleges of the Oratorians at Beaune and then at Lyons, where at the age of sixteen he was made a teacher of physics. It is not improbable that Gaspard Monge had access to a publication of one François Frezier, a French infantry officer, who published in 1738 a treatise on stereotomy as applied to stone cutting and architecture. In this work the advantages to be derived from associating similar scale elevation and plan drawings of the structure—in particular of a cut stone—were at least suggested. For several years prior to his appointment as professor of mathematics in the military school at Mézières in 1768, Gaspard Monge was employed there as a draftsman on the designing of the then popular type of fortifications with their complex outlines both in elevation and plan. While at work on some of the computations he discovered and applied his principle of geometrical solution for some of the so-called problems of defilement. The ordinary mathematical procedures for the solution of these problems were very laborious. When Monge presented his geometrical solutions in which he associated in precise and mathematical fashion several common-scale elevation and plan drawings, there was extreme skepticism of the value and the utility of so simple a method. As soon as it was appreciated that Monge's method constituted a precise mathematical procedure, it was made a military secret and Monge was prohibited from revealing it to any except the students in that particular military school.

Gaspard Monge became the professor of descriptive geometry of the Normal School in Paris in 1795, and later of the École Polytechnique, which was successor to the Normal School. Prior to this he had developed his so-called descriptive geometry from the original discovery of the utility of the intersections of the vertical and horizontal planes on which the various elevation and plan views were made in his first solutions of 1765, through the development of the whole system of descriptive geometry which he was able to make public in 1795 in his lectures. His book on descriptive geometry was first published in 1801. This book might be regarded as the first formal treatise on modern engineering drawing, in that it not only presented the methods to be employed in describing any structure so that all the details of its form might be revealed to anyone who would read it, but also demonstrated that all the problems related to distances, areas, angles, and volumes might be solved graphically.

As one would expect, there have been both quick appreciation and ever-increasing use of the subject of descriptive geometry in the past century of world-wide industrialization. One of Monge's pupils of the École Polytechnique, by name Claude Crozet, was employed in 1816 to be an instructor in descriptive geometry at the United States Military Academy at West Point. In 1821 Crozet published his treatise on descriptive geometry for the use of cadets at the Military Academy. In 1826 Professor Charles Davies, who had succeeded Crozet in the department of mathematics at West Point in 1823, published the first extensive work on descriptive geometry in the English language. Professor Davies was, in turn, succeeded by Professor Albert Church, one of his own pupils, who published his textbook on the elements of descriptive geometry in 1865. Meanwhile another textbook on the subject of descriptive geometry had been published by Professor Warren in 1860. Thus three of the first four American books on descriptive geometry were prepared at our military academy.

The books of Warren and Church sufficed for the needs of engineers in their work for the remainder of the century. Whether anything in particular or of value has been contributed by later authors since 1900 may be questioned. The fact remains, however, that since 1900 the developments in engineering and the requirements resulting from the ever-increasing tempo of design, manufacture, and operation of everything of an engineering nature have been at least such as to make many engineering teachers feel that books on mathematics have needed some adaptation to current practices in engineering. Whether their contentions that such adaptations are needed and are of value are correct or not, a score or more textbooks on descriptive geometry have appeared since 1900, and in most cases emphasis has been placed on the adaptation to modern engineering needs. These books, including this one, are but modifications of the works of Professors Warren and Church, whose books in turn were but modernizations of the original book of Gaspard Monge.

Without attempting to pass on the merits of any of the various modifications presented in present textbooks, it would seem well for us to appreciate that present-day engineering drawing is but a modernized adaptation of descriptive geometry in which are employed all the principles, methods, and devices of descriptive geometry as they were discovered and first employed by Monge in solving his problems of fortifications.

# *PART I*

## CHAPTER I

## ABBREVIATIONS, NOTATION, CONVENTIONS, DEFINITIONS

**1.** It is assumed that any process of learning has as its objective the acquisition of some degree of working mastery of the subject or skill in question. The total process of learning for such a purpose is analyzable into three divisions. The first division is **comprehension** of the subject matter or the manual skill that is to be learned. This division involves the use of books, observation of demonstrations by others, lectures, pictures, models, etc. The second division is **clarification.** Here the learner carefully surveys and discusses step by step what he has read, heard, seen, or done under guidance, to the end of appreciating details, reasons, sequences, and effects. The third division is **fixation,** in which further study, the solution of problems, and the careful supervision of one's own repetitive performance enable the learner to acquire skill, conserve energy, and understand the broader application of principles.

In all the above processes there will be an inevitable need for spoken and written words and for sound and graphical symbols. Only thus do we convey information in the formal fashion. The need for standard identifying sound and graphical symbols to convey information is clearly imperative. How many times one fails to *understand* or *comprehend* for lack of an adequate supply of standard sound and graphical symbols. The finest possible lecture or written discussion is meaningless unless the learner is well acquainted with the set of sound and graphical symbols being used. If, in turn, the learner needs to acquire skill in any process of conveying information, he is equally handicapped unless he can employ standard sound and graphical symbols accurately and without conscious effort.

**2.** For the reasons just discussed it is important that each student make this chapter a constant handbook of reference; for, in the thorough mastery of correct notation and conventions, lies much of the victory in descriptive geometry. Carelessness in these matters, as well as in methods of study and execution of the work, makes a nightmare of indefiniteness and intangibility of descriptive geometry, which is a beautifully precise and readily understandable subject.

It is not desired that any student attempt to memorize the following symbols and abbreviations, but that he refer constantly to them for information on the proper lettering of points, designation of certain angles, and the character of line to be used for each different purpose. In a short time all these will have been memorized from constant use, and descriptive geometry will have been made a far easier subject for the small amount of trouble.

## ABBREVIATIONS

**3.**

$H$ = horizontal plane of projection.
$V$ = vertical plane of projection.
$P$ = profile plane of projection.
$GL$ = ground line.
$G_1L_1$ = profile ground line.

## ILLUSTRATIONS

**4.** It will be observed that most of the illustrations in this edition include *arrows* and *numbers* to assist in following the progress of the solution. The numbers indicate the *order of progress* and the *arrows* indicate the *direction of progress*. While neither are necessary and should be disregarded by those who do not need them, they are particularly desirable for handbook service of the book.

## STANDARD CONVENTIONAL LINES

**5.**

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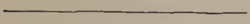
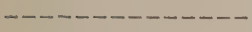
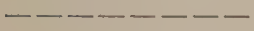
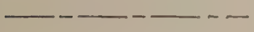
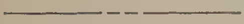
|---|---|---|
|  | Ground lines<br>Given lines<br>Auxiliary lines<br>Construction lines<br>Required lines | *M*<br>*M*<br>*M*<br>*L*<br>*H* |
|  | Projection lines<br>(1/16″ dash, 1/32″ space) | *M* |
|  | Hidden lines<br>(3/16″ dash, 1/16″ space) | *M* |
|  | Traces of given planes<br>Traces of required planes<br>(3/8″ dash, 1/16″ dot and space) | *M*<br>*H* |
|  | Axes of solids<br>Axes of revolution<br>(1/2″ dash, 1/16″ double dot and space) | *M*<br>*M* |

*H*—Heavy *M*—Medium *L*—Light

## SYMBOLS

**6.** $\perp$ = "perpendicular to."

$//$ = "parallel to."

$\diagup$ = "inclined to."

$\angle$ = "angle."

$\times$ = *angle* between two intersecting *lines*.

= *angle* between two intersecting *planes*.

= *angle* between a *line* and a *plane*.

$\Theta, \theta$ = *angle* which any *line* makes with *H*.

= *angle* which any *plane* makes with *H*.

$\Phi, \phi$ = *angle* which any *line* makes with *V*.

= *angle* which any *plane* makes with *V*.

$\Pi, \pi$ = *angle* which any *line* makes with *P*.

= *angle* which any *plane* makes with *P*.

## NOTATIONS

**7.** $a, a', a''$ = $H$, $V$, and $P$ projections of point $A$.
$HT$, $VT$, and $PT$ = traces of plane $T$.
$h, h', h''$ = $H$, $V$, and $P$ projections of the $H$ piercing point of any line.
$v, v', v''$ = $H$, $V$, and $P$ projections of the $V$ piercing point of any line.
$p, p', p''$ = $H$, $V$, and $P$ projections of the piercing point of any line in either the $P$ plane or any given oblique plane.
$A_r, B_r, C_r$ = revolved positions of points.

## SYSTEM OF COORDINATES

**8.** *Three mutually perpendicular planes,* a *horizontal* ($H$) and *two vertical* ($V$ and $P$), Fig. 1, serve as the **coordinate planes** for the description of points, lines, and planes. These planes intersect in the three axes $OX$, $OY$, and $OZ$. The **origin** $O$ is the intersection of the three axes. The $H$ and the $V$ planes make with each other four **quadrants** designated as indicated in Fig. 1.

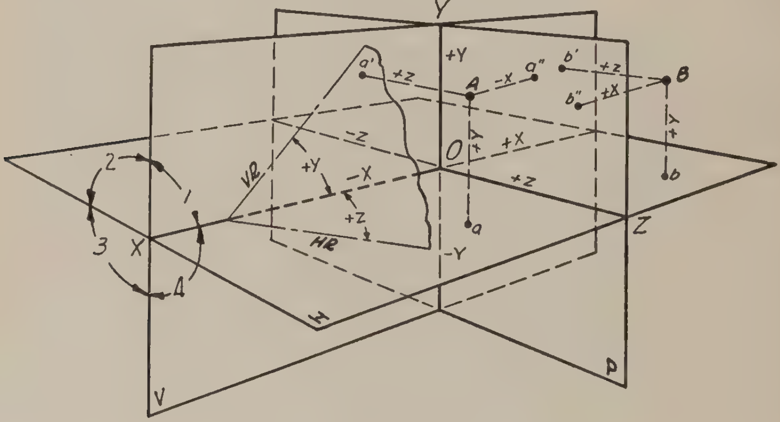


FIG. 1.

The position of a point in space, e.g., $A$ or $B$, Fig. 1, is designated by its distances from the coordinate planes as follows:

$x$ = distance of point to right or left of $P$, as $x$ is + or −.
$y$ = distance of point above or below $H$, as $y$ is + or −.
$z$ = distance of point before or behind $V$, as $z$ is + or −.

The coordinates are always given in this order, and the distances are stated in inches; e.g., the point located by $(6,3,-2\frac{1}{2})$ is 6 inches to the right of the *P* plane, 3 inches above *H*, and $2\frac{1}{2}$ inches behind *V*.

The positions and directions of the *H* and *V* traces of a plane are designated by coordinates and angles, as follows:

$x$ = distance of the point of intersection of the traces with *GL*, to the right or left of the origin, as $x$ is + or −.
$y$ = angle *V* trace makes with *H*, above or below *H*, as $y$ is + or −.
$z$ = angle *H* trace makes with *V*, before or behind *V*, as $z$ is + or −.

Angles $y$ and $z$ are measured from the portion of *GL* to the right of the intersection of the traces.

When a plane is parallel to *GL*, distance $x$ becomes infinite ($\infty$) and angles $y$ and $z$ become 0 degrees. In this case,

$y$ = distance of *V* trace above or below *H*, as $y$ is + or −.
$z$ = distance of *H* trace before or behind *V*, as $z$ is + or −.

## LETTERING ON EXERCISES

**9.** It is important that the projections of all points, lines, and the traces of planes shall be not only lettered, but also lettered carefully. The following alphabets are recommended:

*abcdefghijklmnopqrstu*
*12345 vwxyyz 67890*
*ABCDEFGHIJKLMN*
*OPQRSTUVWXYZ*

abcdefghijklmnopqrstu
12345 vwxyz 67890
ABCDEFGHIJKLMN
OPQRSTUVWXYZ&

## PLANE AND SOLID GEOMETRY "REMINDERS"

### 10. A. Projections

(1) The *projection of a point* upon a plane is the foot of the perpendicular from the point to the plane, Fig. 2.

(2) The *projection of a line* upon a plane is the locus of the projections of all points of the line upon the plane.

(3) The projection upon a plane of a *straight line* not perpendicular to the plane is a straight line, Fig. 3.

(4) The projection upon a plane of a straight line perpendicular to the plane is a point, Fig. 3.

(5) The projection upon a plane of a segment of a straight line parallel to the plane is a segment of a straight line parallel and equal in length to the given segment, Fig. 3.

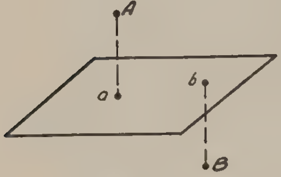


FIG. 2.

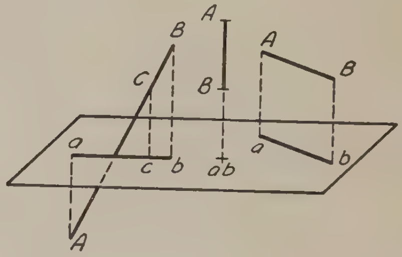


FIG. 3.

### B. Plane

(1) A surface such that a straight line joining any two of its points lies wholly in the surface is called a **plane.**

(2) If two points of a straight line lie in a plane, the whole line lies in that plane.

### C. Coplanar Points and Lines

(1) Points and lines lying in the same plane are said to be **coplanar.**

(2) Lines lying in the same plane must be either parallel or intersecting.

### D. Parallel Lines

(1) Lines that lie in the same plane and cannot meet however far produced are called **parallel lines.**

(2) Through a given external point, one, and only one, line can be drawn parallel to a given line.

(3) Two lines in the same plane are parallel if both are perpendicular to a third line in the same plane.

(4) Two lines parallel to the same line are parallel to each other.

(5) Two lines perpendicular to the same plane are parallel to each other.

**E. Intersecting Lines**

(1) If two lines lie in the same plane and are not parallel, they must intersect.

(2) Two straight lines can intersect in only one point.

**F. Perpendicular Lines, Fig. 4**

(1) If two straight lines intersect at right angles, they are said to be **perpendicular.**

(2) Through any given point on a straight line an infinite number of lines can be drawn perpendicular to the given line.

(3) All the perpendiculars that can be drawn to a given line at a given point on the line lie in a plane which is perpendicular to the given line at the given point.

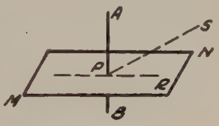


Fig. 4.

(4) From a given external point there can be drawn one, and only one, line perpendicular to a given line.

(5) One, and only one, straight line can be drawn perpendicular to both of two non-parallel non-intersecting lines.

**G. Intersection of Planes**

(1) The intersection of any two surfaces is the locus of all points common to the two surfaces.

(2) If two planes intersect, their intersection is a straight line, Fig. 5.

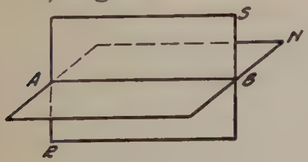


Fig. 5.

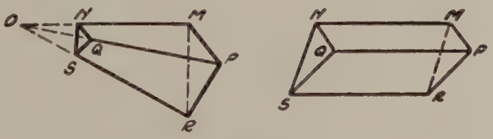


Fig. 6.

(3) If three planes, not passing through the same line, intersect each other, their three lines of intersection are concurrent, or else they are parallel, each to each, Fig. 6.

(4) If a plane intersects two parallel planes, the lines of intersection are parallel, Fig. 7.

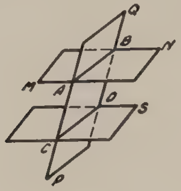


FIG. 7.

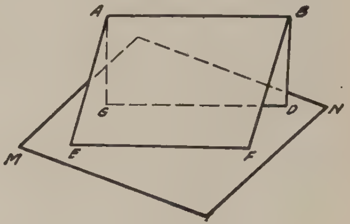


FIG. 8.

### H. Parallel—Line and Plane, Fig. 8

(1) If two straight lines are parallel, a plane containing one of the lines, and only one, is parallel to the other.

(2) A line is parallel to a plane if it is parallel to any line in that plane.

(3) If a straight line is parallel to a plane, the intersection of the plane with any plane passing through the given line is parallel to the given line.

(4) If through any point in a given plane a line is constructed parallel to a line which is parallel to the given plane, the line so constructed will lie in the given plane.

(5) If a line lying in one of two intersecting planes is parallel to the other plane, it is parallel to the line of intersection of the two given planes.

(6) If a line is parallel to a given plane, all points of the line are equally distant from the plane.

(7) If a line is parallel to a given plane, it will never intersect the plane however far produced.

(8) If a line is parallel to a plane, it is parallel and equal to its projection on that plane.

(9) Through any given external point an infinite number of straight lines can be drawn parallel to any given plane.

(10) Through any given external point one, and only one, line can be drawn parallel to both of two intersecting planes; it will be parallel to their line of intersection.

### I. Parallel Planes, Fig. 9

(1) Two planes are said to be **parallel** if they cannot intersect however far produced.

(2) All the straight lines that can be drawn through a given external point parallel to a given plane lie in a plane containing the given point and parallel to the given plane.

(3) If two intersecting lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.

(4) If two planes are parallel, each line in one plane is parallel to the other plane.

(5) If two planes are parallel, all points in one plane are equally distant from the other plane.

(6) If two planes are both perpendicular to the same straight line, they are parallel to each other.

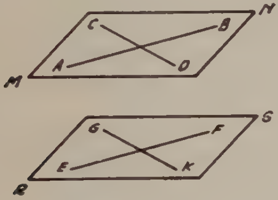


FIG. 9.

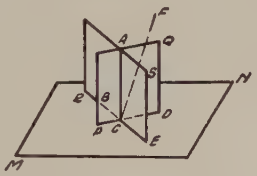


FIG. 10.

**J. Perpendicular—Line and Plane, Fig. 10**

(1) Through any given point one, and only one, line can be drawn perpendicular to a given plane.

(2) If a line is perpendicular to a plane, it is perpendicular to every line in the plane and passing through the foot of the perpendicular.

(3) If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of those lines.

(4) If one of two parallel lines is perpendicular to a plane, the other also is perpendicular to the plane.

(5) If each of two intersecting planes is perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

(6) A plane is perpendicular to a straight line if the line is perpendicular to the plane.

(7) If a plane is perpendicular to a line, then any plane parallel to that plane is perpendicular to the given line.

### K. Perpendicular Planes, Fig. 10.

(1) If a straight line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.

(2) If two planes are perpendicular to each other, any line in one of them, perpendicular to their intersection, is perpendicular to the other.

(3) If a plane is perpendicular to a line lying in another plane, the two planes are perpendicular.

### L. Determining a Plane

(1) A plane is determined by certain lines or points if it contains the given lines or points, and no other plane can contain them.

(2) One plane, and only one, can be passed through two given intersecting straight lines.

- (*a*) A straight line and a point not in the line determine a plane.
- (*b*) Three points not in a straight line determine a plane.
- (*c*) Two parallel lines determine a plane.

(3) A plane is said to be determined by given conditions if that plane and no other plane fulfils those conditions.

- (*a*) Through any given external point an infinite number of planes can be passed parallel to any given straight line.
- (*b*) Through any given external point one, and only one, plane can be passed parallel to any two given (non-parallel) straight lines.
- (*c*) Through any given straight line one, and only one, plane can be passed parallel to any other (non-parallel) straight line.
- (*d*) Through any given external point one, and only one, plane can be passed parallel to any given plane.
- (*e*) No plane can be passed through a given straight line and parallel to a given plane (unless the given line be parallel to the given plane).
- (*f*) Through any given point one, and only one, plane can be passed perpendicular to any given straight line.
- (*g*) No plane can be passed through a given straight line and

perpendicular to any other straight line (unless the two given lines be rectangular).

(*h*) Through any given point an infinite number of planes can be passed perpendicular to any given plane.

(*i*) Through any given point one, and only one, plane can be passed perpendicular to any two given (non-parallel) planes; it will be perpendicular to their line of intersection.

(*j*) Through any given straight line one, and only one, plane can be passed perpendicular to any given plane (unless the line be perpendicular to the given plane).

(*k*) Through any given external point one, and only one, plane can be passed parallel to any given straight line and perpendicular to any given plane (unless the given line be perpendicular to the given plane).

## M. Intersections: Planes—Line and Plane

(1) The straight line of intersection of two planes is the locus of all points common to the two planes.

(2) Being a straight line, the intersection of two planes can be determined either by two of its points or by one point and its direction.

(3) If any line of one plane intersects a line of another plane, the intersection of these lines is a point common to the two planes; hence it is on their line of intersection.

(4) If two intersecting planes are cut by a third plane, the lines cut from the given planes will intersect in a point on their line of intersection, Fig. 6; or—

(5) If two intersecting planes are cut in parallel lines by a third plane, the line of intersection of the two given planes will be parallel to the lines cut from them by the third plane, Fig. 6.

(6) If each of two intersecting planes is perpendicular to a third plane, their line of intersection may be determined by one point and its known direction: perpendicular to the third plane.

(7) If each of two intersecting planes is parallel to a given line, their line of intersection may be determined by one point and its known direction: parallel to the given line.

(8) If any line of one plane pierces another plane, the piercing point is common to the two planes; hence it is on their line of intersection.

### N. Revolution of a Plane

(1) A plane may revolve about any line in it as an axis, and, as it does so revolve, it can contain any particular point in space in one, and only one, position.

(2) If one of two intersecting planes is revolved about their line of intersection as an axis, in one and only one position will it coincide with the other plane.

### O. Angle—Line and Plane

(1) The angle that a line makes with its projection on a plane is considered to be the angle that it makes with the plane, and is called the **inclination** of the line to the plane.

(2) The acute angle that a straight line makes with its own projection upon a plane is the least angle that it makes with any line of the plane.

### P. Dihedral Angle, Fig. 11

(1) The opening between two intersecting planes is called a **dihedral angle.** The planes forming a dihedral angle are called its faces, and their intersection is called the edge of the angle.

(2) The size of a dihedral angle is a measure of the amount of rotation necessary to bring one of the plane faces into coincidence with the other.

(3) The plane angle formed by two straight lines, one in each plane, perpendicular to their intersection at the same point, is called the "**plane angle of the dihedral angle.**"

(4) The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.

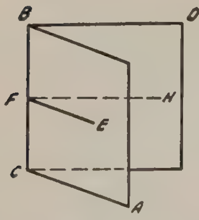


Fig. 11.

(5) The angle between two planes is the angle measured between the two lines cut from those planes by a third plane passed perpendicular to their line of intersection at any point on the intersection.

## DEFINITIONS

**11.**

| | ARTS. |
|---|---|
| Alphabet of a line | 25 |
| Alphabet of a plane | 38 |
| Alphabet of a point | 21 |
| Apex | 105 |
| Axis | 103 |
| minor and major | 109 |
| Base | 103 |
| Brilliant line | 148 |
| Brilliant point | 148 |
| Cone ($\theta$, $\phi$, $\pi$) | 107 |
| Cones | 105 |
| Convolutes | 102 |
| Coordinate planes | 14 |
| Coplanar | 34 |
| Cylinders | 103 |
| Descriptive geometry | 12 |
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| Direct light | 141 |
| Element | 101 |
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| Line of shadow | 143 |
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| Parallelism | 32 |
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| Visual plane | 155 |
| Visual ray | 155 |

## CHAPTER II

# THE POINT AND LINE

**12. Descriptive geometry** is that division of geometry which enables the engineer:

1. To so describe by drawings any object or project involving form that the description will reveal all the essential engineering facts.
2. To determine from the above description any and all desired facts as to distances, angles, intersections, areas, and volumes.
3. To make from the above description any constructions required to introduce given distances, angles, intersections, areas, or volumes.

The purposes of descriptive geometry are accomplished through the employment of four principles:

*A*. The Principle of Orthogonal Projection.
*B*. The Principle of Perpendicularity.
*C*. The Principle of Intersection.
*D*. The Principle of Revolution.

## PRINCIPLE "A"

## ORTHOGONAL PROJECTION

**13.** The engineering problems of geometry are related to three-dimensional projects or objects. They can be solved graphically, however, only through the medium of plane or two-dimensional geometry on the drawing board. Such a principle of description must therefore be employed as will reduce the three-dimensional problems to a two-dimensional status. It is obvious then that the process of description must produce a group of related plane-geometry drawings, in each of which but two dimensions are represented; but all dimensions are represented in the entire group.

The required individual plane-geometry descriptions are called **"projections"** and are made upon a set of conventional intersecting planes, usually three, a *horizontal* and *two vertical, at right angles to each other*, Fig. 1. The *projections are made upon these planes by means of lines through all significant points* of any structure and *perpendicular to the respective planes of description.* This principle of description is the principle of right angle or **orthogonal projection.** When made according to this principle the descriptions will support any and all desired precise calculations of values and determination of intersections and shapes.

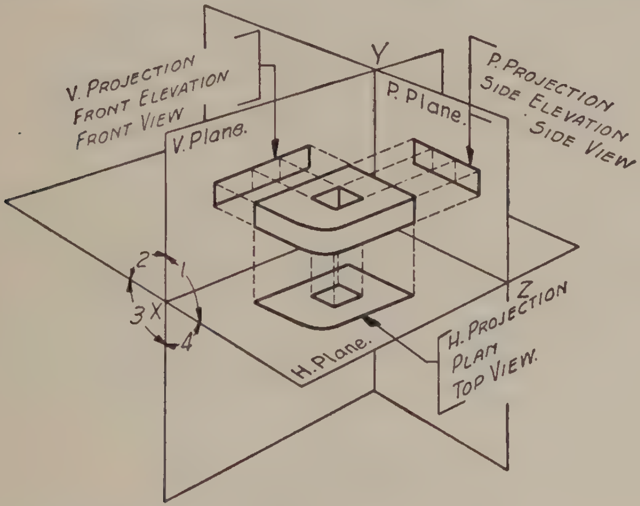


FIG. 1.

## COORDINATE PLANES OR PLANES OF PROJECTION

**14.** The three conventional planes mentioned in the preceding article are named the **planes of projection,** and they are: *two vertical planes,* which are perpendicular to each other, and a *horizontal plane,* Fig. 1. These same planes are usually used as coordinate planes in stating the location of a point in space.

For convenience, we speak of one of the *vertical planes* as the

**$V$ plane,** the other as the **profile** or **$P$ plane.** The *horizontal plane* is called the **$H$ plane.** These three planes are shown in their relative positions in Fig. 1.

## GROUND LINE

**15.** The line of intersection of the $H$ and $V$ planes is known as the **ground line** and for convenience is referred to as $GL$. The intersection of the $P$ plane with the $V$ plane is known as the **profile ground line** and is referred to as $G_1L_1$.

The $H$ and $V$ planes form four dihedral angles or quadrants, which are named the first, second, third, and fourth in the order shown in Fig. 1.

## PROJECTIONS—VIEWS

**16.** The *representations* or *descriptions* of points, lines, and objects on the *planes* of *projection* are named **projections, views,** or **plan** and **elevations** as indicated in Fig. 1. Such representations are made according to the precise mathematical principles of orthogonal projection.

## PROJECTING LINES

**17.** The *$H$ projection of a point,* for example, $A$, Fig. 2, *is the point in which the orthogonal line of sight through $A$* (a vertical line) *pierces the $H$ plane.* The line of sight is named a **projecting line.** This line, as all lines, is indefinite in extent, but there is no good reason for describing any but the portion of it between the point and the plane. The projecting lines of a point may then be defined as the lines passed through the point perpendicular to the planes of projection. $Aa$, $Aa'$, and $Aa''$, Fig. 2, are projecting lines.

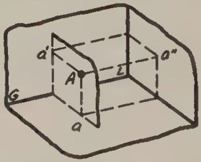


FIG. 2.

## PROJECTIONS OF A POINT

**18.** In summary of the preceding several articles, the *H projection of a point, A, is the H piercing point of a perpendicular through A to the H plane; the V projection is the V piercing point of a perpendicular through A to the V plane;* and the *P projection is the P piercing point of a perpendicular through A to the P plane.* These projections are designated, as in Fig. 2: the $H$ projection $a$, $V$ projection $a'$, and $P$ projection $a''$. The projections of all points are designated or lettered in a similar manner.

## POINT DETERMINED BY ANY TWO OF ITS PROJECTIONS

**19.** If through the $H$ projection, $a$, of a point, $A$, Fig. 2, a perpendicular is erected to the $H$ plane, it will contain the point $A$. A perpendicular to the $V$ plane through the $V$ projection, $a'$, will also contain point $A$. The point is hence determined, being the intersection of these two perpendiculars.

In similar fashion perpendiculars to $P$ through $a''$ and to $V$ through $a'$ or to $H$ through $a$ determine the point $A$ in space.

## LOCATION OF POINTS

**20.** If through the projecting lines $Aa'$ and $Aa$, Fig. 2, a plane is passed, it will be perpendicular to both $H$ and $V$ and intersect $H$ and $V$ in lines perpendicular to $GL$. These lines with $Aa'$ and $Aa$ form a rectangle in which the distance from $a'$ to $GL$ equals $Aa$ and the distance from $a$ to $GL$ equals $Aa'$. From the above we deduce that:

(*a*) The distance from a point to the $H$ plane is shown by the distance from its $V$ projection to $GL$.

(*b*) The distance from a point to the $V$ plane is shown by the distance from its $H$ projection to $GL$.

(*c*) If a point lies in $V$, its $H$ projection is in $GL$; it cannot be in $GL$ otherwise. Conversely, if the $H$ projection of a point lies in $GL$, the point lies in $V$.

(*d*) If a point lies in $H$, its $V$ projection is in $GL$; it cannot be in $GL$ otherwise. Conversely, if the $V$ projection of a point lies in $GL$, the point lies in $H$.

### ALPHABET OF A POINT

**21.** A point may be placed in nine different positions with respect to the planes of projection; these nine positions constitute what is known as the **alphabet of a point,** Fig. 3.

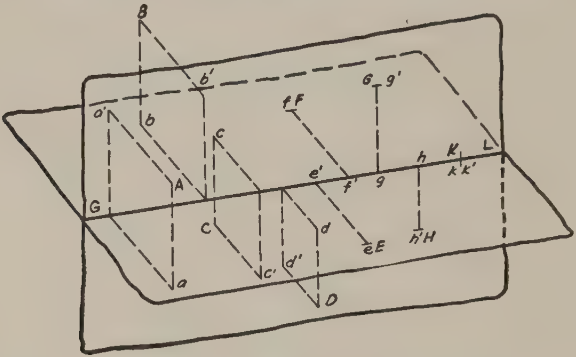


FIG. 3.

1. In the first quadrant.
2. In the second quadrant.
3. In the third quadrant.
4. In the fourth quadrant.
5. In the *H* plane before *V*.
6. In the *H* plane behind *V*.
7. In the *V* plane above *H*.
8. In the *V* plane below *H*.
9. In both *H* and *V*.

### REVOLUTION OF THE COORDINATE PLANES

**22.** The question as to how the projections on the three planes may be represented on a plane drawing must now be considered. This is accomplished through the revolution of the *H* and *P* planes about their respective ground lines into coincidence with *V*. With *GL*, the line of intersection of *H* and *V*,

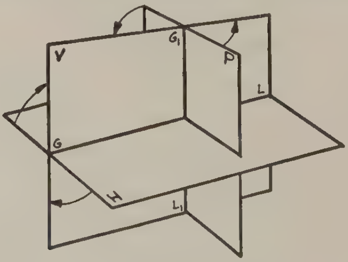


FIG. 4.

Fig. 4, as an axis, the part of *H* in front of *V* is revolved *downward,* the part behind *upward,* of course, until *H* coincides with

*V*. With $G_1L_1$ as an axis, the portion of *P* in *front* of *V* is revolved to the *right* (or *left*), depending upon whether the object described is in the first or third quadrant and also upon whether the profile plane is to the right or left of the object. The portion *behind* is revolved to the *left* (or *right*), until *P* coincides with *V*. The three planes now coincide.

In the revolution just explained, the *V* projections of points are not affected; i.e., the *V* projections of points in the *first* and *second quadrants* are *above GL*, and those of points in the *third* and *fourth quadrants* are *below GL*, regardless of any revolution of planes; the *H* projections of all points in *front* of *V*, however, either in the first or fourth quadrants, being in the portion of *H* which was revolved *downward*, now come *below GL*; see Fig. 5. The *H* projections of all points *behind V*, in either the second or third quadrants, are found *above GL* since that part of *H* was revolved upward.

As the *H* plane is revolved about *GL*, as an axis, the *H* projection of a point moves in a plane which is perpendicular to *GL*; after the revolution the *H* projection is therefore found on the perpendicular to *GL*, through the *V* projection of the point. *The H and V projections of a point are always on the same perpendicular to GL.*

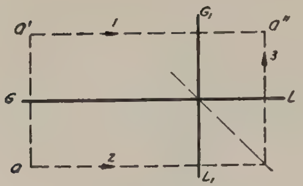
After the revolution of the *P* plane into coincidence with *V*, the *P projection will lie on a horizontal line through the V projection* and as far to the right or left of $G_1L_1$ as the point is in *front* of or *behind V*.

The orthogonal projections of the nine positions which constitute the *alphabet of a point* are shown in Fig. 5. (Compare with Fig. 3.)

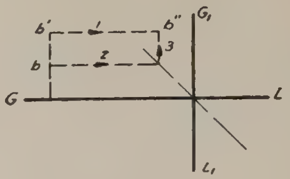
See Test Exercise 1, page 228.

## LINES

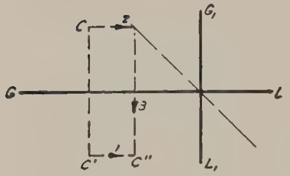
**23.** Since a straight line is generated by a point moving in some fixed direction, the *H projection of a straight line is made up of the H projections of the successive positions of the generating point.* The *H* projecting lines of all these points, being perpendicular to *H*, constitute a plane which is perpendicular to *H*; the line of intersection of this plane with *H* is the *H* projection of the given line. The *H projection of a line* may thus be considered



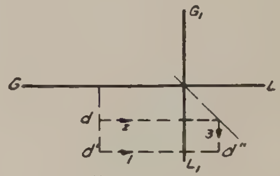
1ST QUADRANT



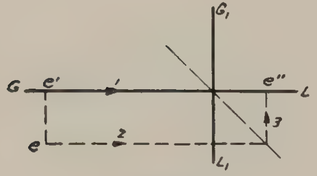
2ND QUADRANT



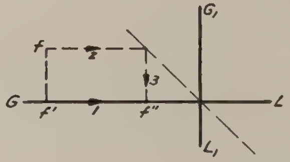
3RD QUADRANT



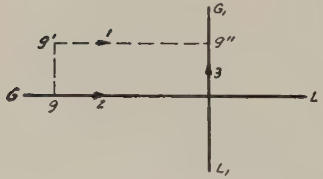
4TH QUADRANT



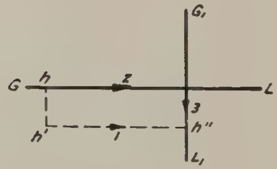
IN H BEFORE V



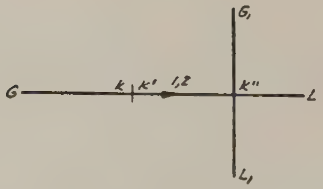
IN H BEHIND V



IN V ABOVE H



IN V BELOW H



IN H AND V

FIG. 5.

the *line of intersection with H of a plane passing through the given line and perpendicular to H.* This auxiliary plane is named the ***H* projecting plane** of the line. The fact that this projecting plane is perpendicular to *H* makes it a *vertical* plane, and the *H* projection of its intersection with *H* (the *H* projection of the line) is the same regardless of the level at which the *H* plane may be located. The *V* and *P* projections of a straight line are the lines of intersection with *V* and *P* of projecting planes through the line and perpendicular respectively to the *V* and *P* coordinate planes.

Since the *H* projection of a straight line is a straight line, the projection may be determined by passing perpendiculars to *H* through any two points of the line; the line joining the *H* piercing points of these perpendiculars, Fig. 6, is the required *H* projection. The *V* and *P* projections may be determined in a similar manner.

Hereafter in this book it may be assumed that the word "*line*" implies a *straight* (or *right*) line unless otherwise designated. Since any two points determine a straight line, a line is usually designated by its two extremities if it has a fixed length. Otherwise any two points of the line may be chosen at random for the purpose of locating the entire line in any other view. In general, the line is read from left to right. Lines in a profile plane are read from front to rear, and, if vertical, are read downward.

### LOCATION OF A LINE

**24.** If two projections of a line are given, the line is in general completely determined. For, if through the projections of any point of a line, e.g., *b* and *b′*, Fig. 6, perpendiculars to the planes of projection are erected, these perpendiculars will intersect in the corresponding point, i.e., *B*, of the straight line. The point *A* is similarly determined from *a* and *a′*. The straight line joining these two points, *A* and *B*, is the required line, Fig. 6. Or again, if through the *H* projection a plane is passed perpendicular to *H*, and through the

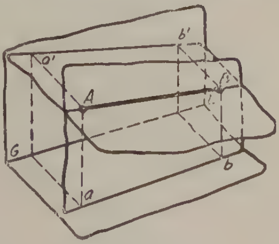


FIG. 6.

*V* projection a plane perpendicular to *V*, the line will lie in both these planes; hence, is their line of intersection.

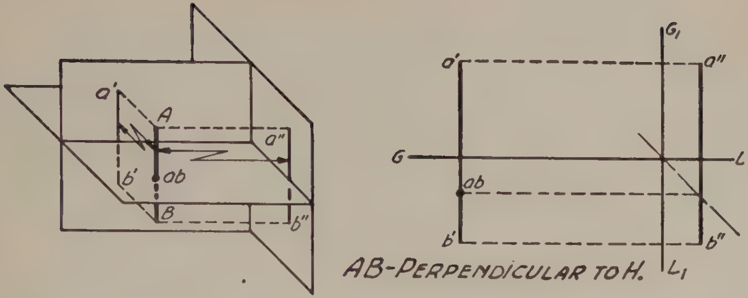
## ALPHABET OF A LINE

**25.** In determining the different positions in which a line may be placed with respect to the planes of projection, the quadrants are not considered and the lines are understood to be indefinite in extent. The several positions constituting the **alphabet** are the *characteristic directional* positions in which the line may be placed. These positions are shown pictorially and in orthogonal projection in Figs. 7(*a*) and 7(*b*).

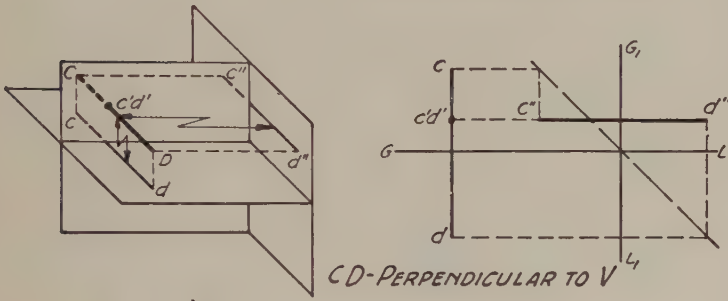
These positions of the line and corresponding projections are as follows:

| | |
|---|---|
| 1. Perpendicular to *H* | *V* projection a line perpendicular to *GL*.<br>*H* projection a point. |
| 2. Perpendicular to *V* | *V* projection a point.<br>*H* projection a line perpendicular to *GL*. |
| 3. Parallel to *P* and inclined to *H* and *V* | Both projections perpendicular to *GL*. |
| 4. Parallel to *H* and inclined to *V* | *V* projection parallel to *GL*.<br>*H* projection inclined to *GL*. |
| 5. Parallel to *V* and inclined to *H* | *V* projection inclined to *GL*.<br>*H* projection parallel to *GL*. |
| 6. Parallel to both *H* and *V* | Both projections parallel to *GL*. |
| 7. Inclined to *H*, *V*, and *P* | Both projections inclined to *GL*. |

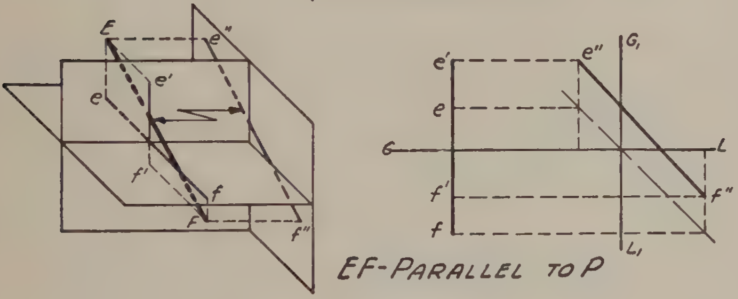
No mention is made here of the profile relations, but they may be easily understood from the figures.



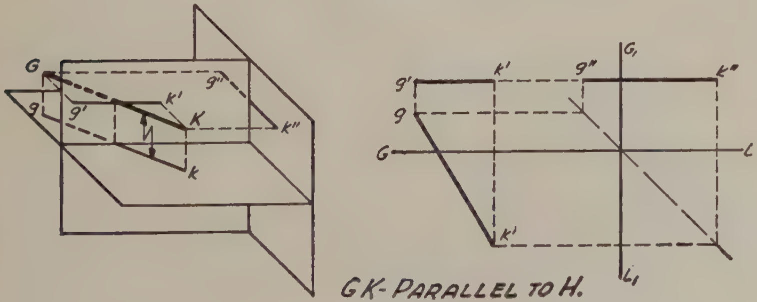
AB-PERPENDICULAR TO H.



CD-PERPENDICULAR TO V

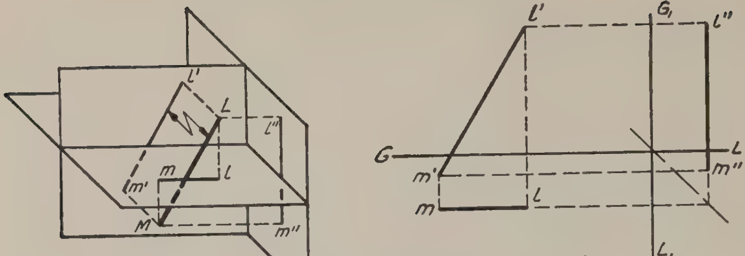


EF-PARALLEL TO P

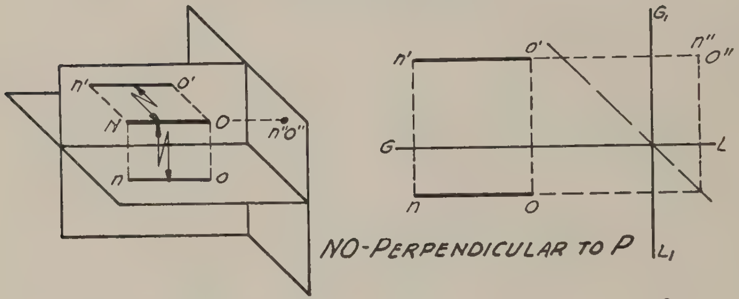


GK-PARALLEL TO H.

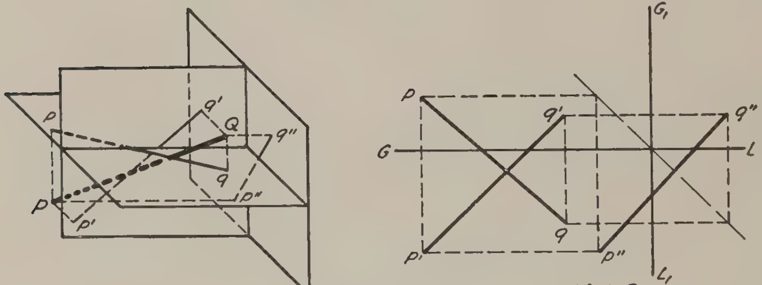
FIG. 7(a).



LM-PARALLEL TO V



NO-PERPENDICULAR TO P



PQ-INCLINED TO H. V. & P.

FIG. 7(b).

## PROFILE PROJECTIONS OF POINTS AND LINES

**26.** When dealing with the *profile projections* of points and lines, the *profile plane* is considered as revolved about $G_1L_1$ until it coincides with $V$. In this view $G_1L_1$ represents the intersection of $P$ and $V$, and $GL$ represents the intersection of $P$ and $H$; or, for convenience, $G_1L_1$ may be considered the $V$ plane, and $GL$, the $H$ plane, viewed edgewise.

The $P$ *projection* of a point $A$ is obtained by passing a projecting line through $A$ perpendicular to $P$; the piercing point of this perpendicular in $P$, Fig. 2 [Article 18], is the $P$ projection of $A$. This projecting line pierces $P$ at a distance above $H$ equal to $Aa$, which in turn equals the distance from $a'$ to $GL$. Likewise, it pierces $P$ at a distance before $V$ equal to $Aa'$ which is equal to the distance from the $H$ projection, $a$, to $GL$ Then, after revolving the profile plane about $G_1L_1$ into $V$, Fig. 5, $a''$ is found above $GL$ a distance equal to the distance from $a'$ to $GL$, or on a line parallel to $GL$ through $a'$; likewise, since $A$ is in the first quadrant, $a''$ will be to the right of $G_1L_1$, a distance equal to the distance that the $H$ projection, $a$, is below $GL$. The construction for finding the profile projections of points in the various quadrants, from their $H$ and $V$ projections, is shown in Fig. 5. In each case the portion of the $P$ plane in front of $V$ has been revolved to the *right*, and the portion *behind* $V$ to the *left* of $G_1L_1$.

The profile projection of a line joining two points is the line joining the profile projections of those two points.

See Test Exercises 2 and 3, pages 229 and 230.

**27.** Problem 1. **To determine the projections of a line which passes through or contains a given point.**

**Analysis:**

The projections of a line are made up of the projections of all points of the line.

The projections of a point which lies on a line must therefore be found on the corresponding projections of the line.

**Solution:**

Two projections drawn from or through the respective projections of the given point to any other given or assumed point describe the required line.

**Construction:**

Given: Point $A$, Fig. 8.

Required: Any line through point $A$.

Point $B$ is assumed in any quadrant, e.g., the fourth, and at any distances from $H$ and $V$ (*1*,*3*). The corresponding projections of $A$ and $B$ are then connected (*4*,*5*). The projections $a'b'$ and $ab$ describe a line, $AB$, which passes through point $A$

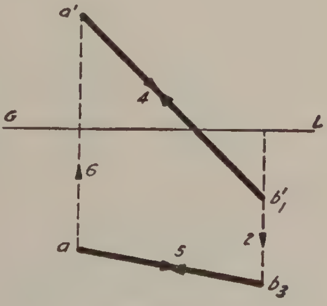


Fig. 8.

since *A* is a point of the line. An alternative procedure is to draw the required projections through *a′* and *a* and in any directions such that the projections of a point, *B*, may be assumed on them.

28. Problem 2. **To determine the projections of a specified line.**

**Analysis:**

A line is defined by any two points on it.

The lines joining the respective projections of two points constitute the projections of the line through those two points.

All the qualities of a line are either directly defined by or can be determined from its projections. These include length, and relative locations and directions.

**Solution:**

If the projections of two points, *A* and *B*, are assumed such that the lines joining their respective projections describe a line of the required qualities, the line *AB* then constitutes the required line.

**Construction:**

Required: To determine a line parallel to *H*, oblique to *V*, extending from a point, *A*, in the first quadrant 1 inch above *H* to a point, *B*, in the second quadrant, Fig. 9.

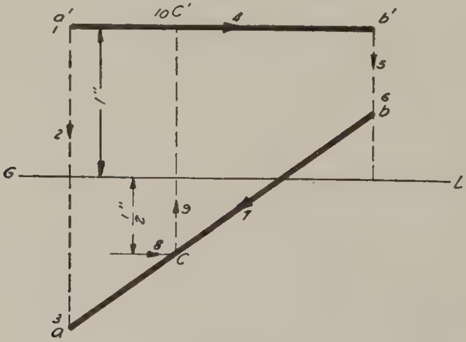


Fig. 9.

The *V* projection of a point in the first quadrant 1 inch above *H* is 1 inch above *GL*; *a′* is hence assumed 1 inch above *GL* and *a* vertically below it and at any distance below *GL* (*1,3*).

The $V$ projection of the line is horizontal and both projections of $B$ are above $GL$ since $B$ is in the second quadrant (*4,6*).

29. Problem 3. **To determine the projections of any particular point on a line.**

**Analysis:**

If a point lies on a given line in space, its projections lie on the corresponding projections of the line and are vertically above and below each other [Article 22].

The location of a point with reference to the coordinate planes is indicated by the locations of its projections with reference to the ground lines.

**Construction:**

Given: Line $AB$, Fig. 9.

Required: A point $C$ on $AB$ and $\frac{1}{2}$ inch in front of $V$.

The $H$ projection of point $C$ lies on the $H$ projection of $AB$ and is $\frac{1}{2}$ inch below $GL$ (*8*). The $V$ projection is on $a'b'$ and vertically above $c$ (*9,10*).

30. Problem 4. **To find the *H, V,* and *P* piercing points of a given line.**

**Analysis:**

(This problem assumes that $H$, $V$, and $P$ are definitely located by $GL$ and $G_1L_1$. The beginner in descriptive geometry is advised to designate the $H$, $V$, and $P$ piercing points of all lines as points $H$, $V$, and $P$.)

The $H$, $V$, and $P$ piercing points of a line are the points which are respectively common to the line and to the $H$, $V$, and $P$ planes.

The $V$ projections of all points in $H$ lie on $GL$; and the $V$ projections of all points of a given line lie on the $V$ projection of the line. The $V$ projection of the point common to a given line and $H$ is therefore the intersection of the $V$ projection of the line with $GL$.

**Solution:**

The intersection of the $H$ and $V$ projections of a given line with $GL$ are respectively the $H$ projection of the $V$ piercing point and the $V$ projection of the $H$ piercing point.

The intersections of vertical lines through these points with the other projections of the line are the other projections of the desired piercing points.

**Construction** (*a*):

Given: Line $AB$, Fig. 10.

Required: $H$ and $V$ piercing points of $AB$.

The intersection of $a'b'$, extended, and $GL$ gives $h'$, the $V$ projection of the $H$ piercing point (*1*). A perpendicular to $GL$ at $h'$ intersects $ab$ in $h$, the $H$ projection of the $H$ piercing point (*2,3*). $ab$, extended, intersects $GL$ at $v$, the $H$ projection of the $V$ piercing point (*4*); the intersection of a perpendicular to $GL$ at $v$ with $a'b'$ locates $v'$, the $V$ projection of the $V$ piercing point (*5,6*).

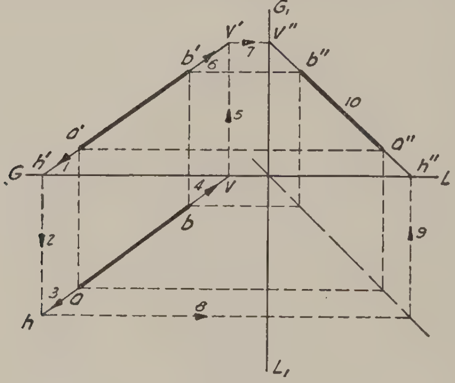


Fig 10.

**Construction** (*b*):

Required: The $P$ piercing point of line $AB$, Fig. 11.

The $V$ projection $p'$, of the $P$ piercing point, is common to the $V$ projection of the line and $G_1L_1$ (*1*). The $H$ projection $p$ is common to the $H$ projection of the line and $G_1L_1$ (*2*). The $P$ projection of the $P$ piercing point is $p''$ (*3,5*).

**Construction** (*c*):

Required: The $H$ and $V$ piercing points of line $AB$, Fig. 12.

The $H$ and $V$ piercing points of lines whose projections are perpendicular to $GL$ may be determined by referring the line to the profile plane. The intersections of the profile projection with $GL$ and $G_1L_1$ are the profile projections of the $H$ and $V$ piercing points.

The profile projection of $AB$ is $a''b''$ (*1,6*). $a''b''$, extended, intersects $G_1L_1$ in $v''$, from which the required projections $v'$ and $v$ are then obtained (*7,11*). $a''b''$, extended, intersects $GL$ in $h''$, from which are obtained the required projections $h'$ and $h$ (*12,18*).

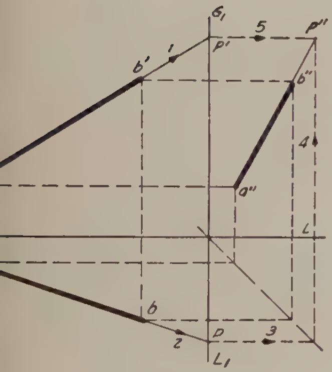


FIG. 11.

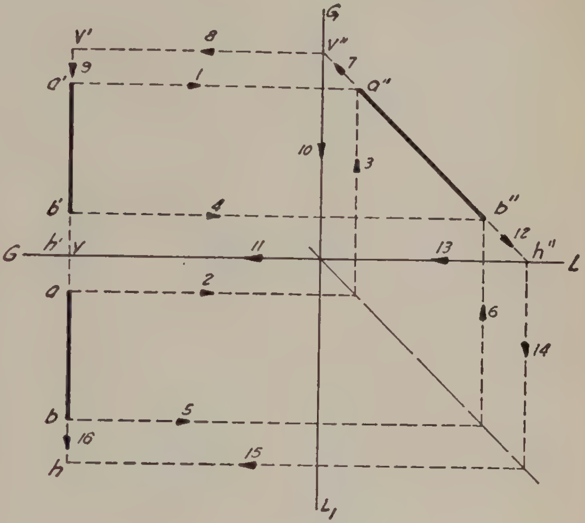


FIG. 12.

See Test Exercises 4 and 5, pages 231 and 232.

**31.** Problem 5. **To assume a line intersecting a given line.**

**Analysis:**

Two lines are said to intersect if they have one point in common.

The projections of the common point must lie on the corresponding projections of both lines, hence at their intersection.

Two lines described by projections are known to intersect if the respective projections intersect in points vertically above and below each other ($H$ and $V$) or on the same horizontal line ($V$ and $P$).

**Solution:**

If a point, *O*, is assumed on a given line and the projections of a second line, *CD*, are drawn through the corresponding projections of the assumed point, the two lines so described intersect.

**Construction:**

Given: Line *AB*, Fig. 13.

Required: A line, *CD*, intersecting *AB* at *O*.

Point *O* is assumed on line *AB* (*1*,*2*). The vertical projection *c′d′* is then drawn through *o′* in any direction (*3*). The *H* projection of *C* is chosen wherever desired on the vertical line through *c′* (*4*). From *c* the *H* projection of *CD* is drawn through *o* (*5*) and the *H* projection of *D* located by its intersection with the vertical line from *d′* (*6*). *AB* and *CD* are intersecting lines because they have point *O* in common.

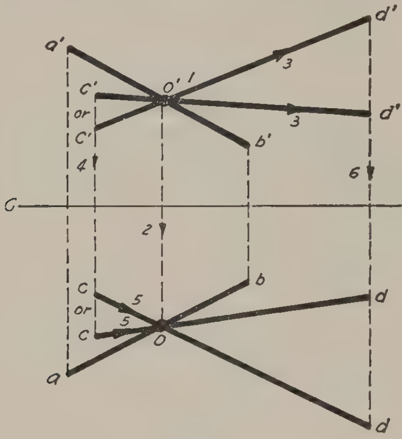
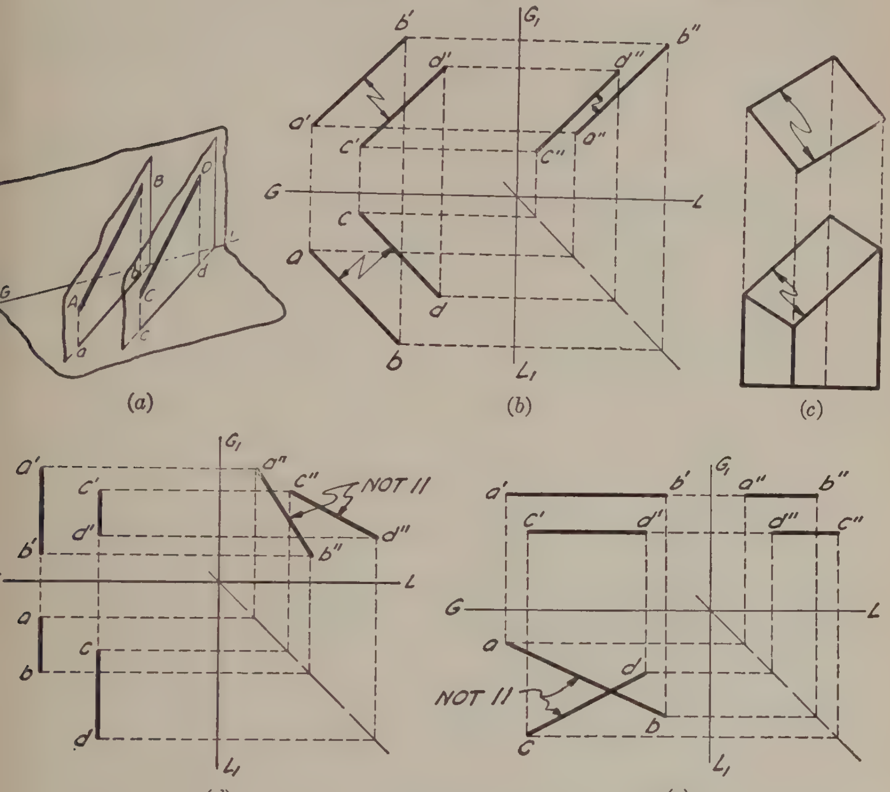


Fig. 13.

See Test Exercises 6 and 7, pages 233 and 234.

## PARALLEL LINES

**32.** If two lines lie in the same plane and do not meet however far extended, they are said to be parallel. If two lines are parallel, their projections on any plane are parallel. Lines *AB* and *CD*, Figs. 14 (*a*-*b*), are given parallel. The *H* projecting planes through these two parallel lines are parallel since the one contains two intersecting lines, *AB* and *Bb*, parallel respectively to the two intersecting lines $CD$ and $Dd$ of the other. If these two parallel planes are intersected by a third plane (the $H$ plane), the lines of intersection, $ab$ and $cd$, are parallel. By



(a) (b) (c)

(d) (e)

FIG. 14.

similar proof the $V$ projections $a'b'$ and $c'd'$ and the profile projections $a''b''$ and $c''d''$ are respectively parallel.

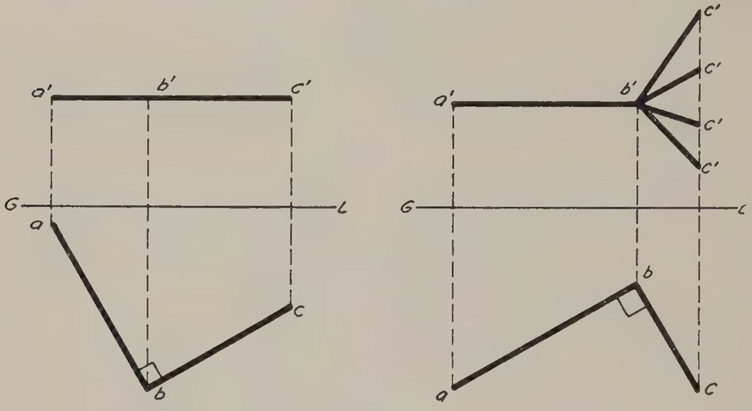
See Test Exercises 8 and 9, pages 235 and 236.

## PERPENDICULAR LINES

**33.** If two lines intersect and make an angle of 90 degrees with each other, they are said to be *perpendicular*.

If two intersecting lines are parallel to a coordinate plane, e.g., $H$, Fig. 15($a$), the angle which their $H$ projections make with each other is the same as the angle between the lines themselves. If two perpendicular lines are both parallel to the same coordinate plane, the projections on that plane will therefore be perpendicular to each other. *These two facts (a) that the two lines are parallel to H and (b) that the H projections are perpendicular to each other, constitute proof that the lines themselves are perpendicular to each other.*

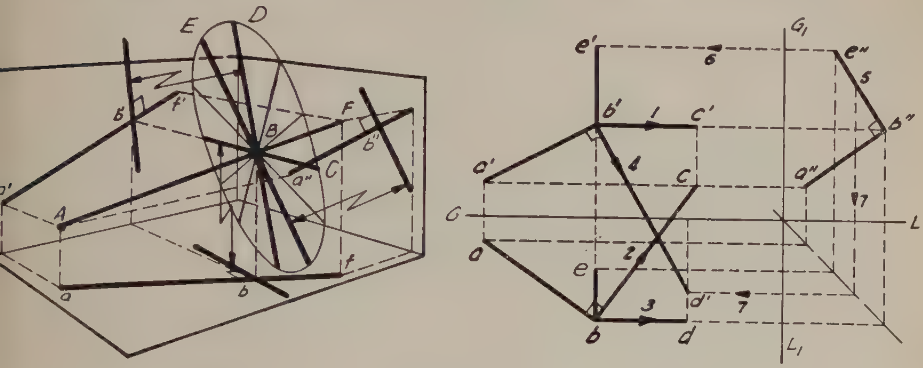
*If one of two perpendicular lines is parallel to a coordinate plane, e.g., H, Fig. 15(b), the H projections of the lines will be perpendicular to each other.* This is because all lines perpendicular to a given horizontal line and passing through a given point of it define a vertical plane perpendicular to the line. The *H* projections of all lines in this plane coincide in the intersection of the plane with *H*, which intersection is perpendicular to the *H* projection of the given line. *The facts (a) that one line is parallel to a coordinate plane and (b) that the projections on that plane are perpendicular to each other constitute proof that the lines themselves are perpendicular to each other.*



(*a*) (*b*)

FIG. 15.

Through any point, *B*, on a given line, *AF*, Fig. 16(*a*), an infinite number of perpendiculars to *AB* can be passed. They lie in a plane through *B* perpendicular to *AF*. One of these lines, *BC*, will be horizontal (*1,2*), Fig. 16(*b*). Another, *BD*, will be parallel to *V* (*3,4*), and a third, *BE*, will be parallel to *P* (*5,7*). The projections of these three lines can always be drawn, e.g., *BC* with its *V* projection *b′c′* horizontal and its *H* projection *bc* perpendicular to the *H* projection of *AB*. Any two of these three perpendiculars (*1*,*2*), (*3*,*4*), or (*5*,*7*) will determine the plane through *B* perpendicular to *AB* and any other line in this plane and through point *B* will be perpendicular to *AB*.



(*a*) (*b*)

Fig. 16.

Through any external point only one perpendicular can be drawn to a given line. The angle of 90 degrees which these lines make with each other will not *show* on the coordinate planes, however, *unless one of them is parallel to a coordinate plane;* so the construction of such a line must await the development of the process of passing a plane through a point perpendicular to a line and finding the point in which the line pierces that plane.

See Test Exercises 10 and 11, pages 237 and 238.

## CHAPTER III

# PLANES

### DEFINED AND DESCRIBED

**34.** A plane is *defined by any two lines in it,* and is completely *described by the projections* of these lines on the coordinate planes. All lines lying in a given plane are called **coplanar** lines. The significant qualities of such lines are that they either *intersect* or are *parallel.* Since every plane must intersect at least two of the coordinate planes, these intersections may serve as the *any two lines,* and their use, for a time at least, serves to simplify some discussions of the simple relations of points, lines, and planes and later processes of determination of values. For these reasons, therefore, we shall give such lines of intersection of planes with the coordinate planes identifying sound and graphical symbols. The sound symbol or name for the intersections with *H*, *V*, and *P* is **traces**, and the graphical symbol is a dash and dot or long and short dash line. Each individual plane must also have a name, a single letter, as *R*, or the letters representing three points, as *ABC*. If the plane is described by its traces, they may be designated as in Figs. 1 and 2, *HR*, *VR*, and *PR*.

### THE *H* AND *V* TRACES OF A PLANE INTERSECT ON *GL*

**35.** If a plane such as *R*, Fig. 1, is oblique to *GL*, it will intersect *GL*; this point of intersection is common to *V* and *R*, hence lies on their line of intersection, *VR*; it is also common to *H* and *R*, hence lies on their line of intersection, *HR*. The point in which *R* intersects *GL* has thus been proved common to *VR*, *HR*, and *GL*; that is, the *H* and *V* traces of plane *R* which is oblique to *GL* must intersect on *GL*. These traces appear as shown on Fig. 2 when the coordinate planes have been revolved.

## LOCATION OF A PLANE

**36.** A plane is definitely determined both as to direction and location by any two lines in it; that is, by any two traces or by any other two lines; for through two intersecting or parallel lines one and only one plane can be passed.

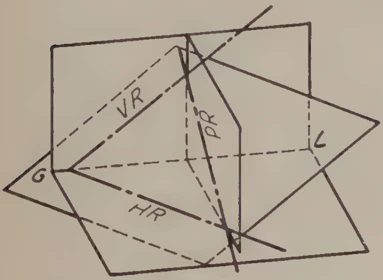


FIG. 1.

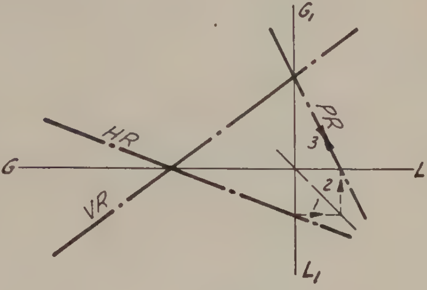


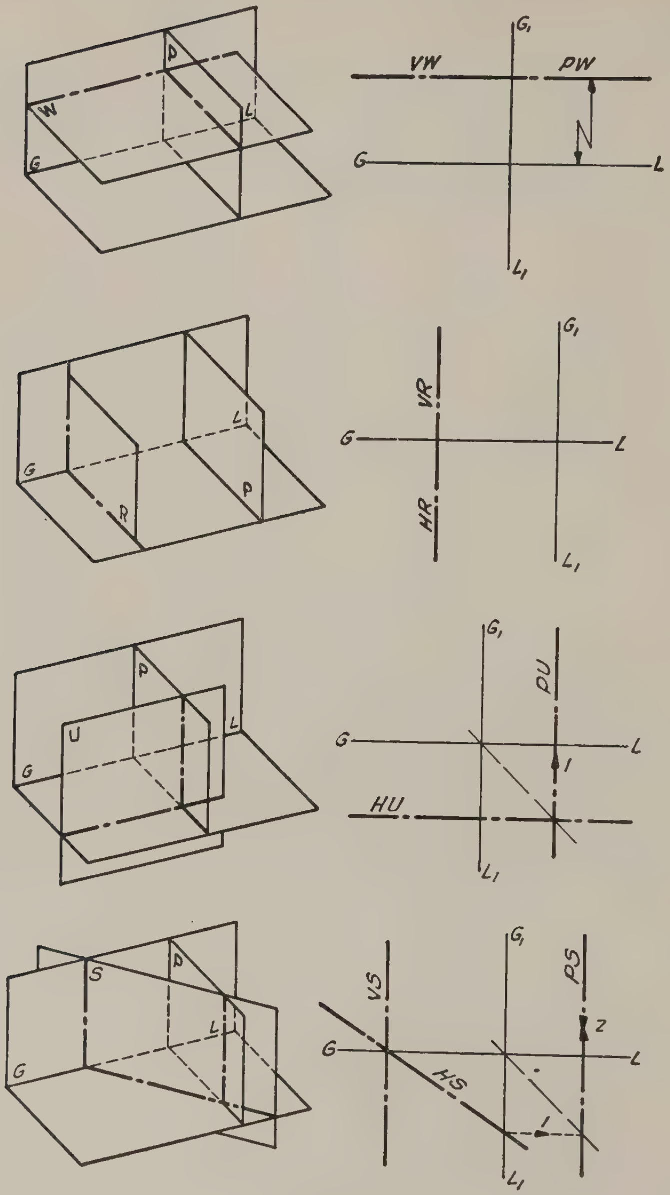
FIG. 2.

## THE PROFILE TRACE

**37.** The profile trace of a plane is its line of intersection with $P$. Since this trace cannot be parallel to the $V$ trace unless the plane is perpendicular to $H$, nor to the $H$ trace unless the plane is perpendicular to $V$, it must intersect them and of course in the points in which those traces pierce $P$, Fig. 1. Since the profile plane is revolved into coincidence with $V$ about its intersection $G_1L_1$, with $V$, the intersection of the $P$ and $V$ traces on $G_1L_1$, Fig. 2, remains fixed. Since $H$ and $P$ are revolved into coincidence with $V$ about different lines in $V$, the $P$ projection of the $P$ piercing point of the $H$ trace, Figs. 1 and 2, is revolved into coincidence with $GL$, while the $H$ projection of this same piercing point is revolved into coincidence with $G_1L_1$.

## ALPHABET OF A PLANE

**38.** The seven different characteristic directions in which planes may be conceived according to their relation to the planes of projection are named the **alphabet of a plane.** These positions are shown both in perspective and in orthogonal projection in Fig. 3. Their relations to the planes of projection,


P
W
L
G
$G_1$
VW
PW
G
L
$L_1$
L
G
R
P
$G_1$
VR
G
L
HR
$L_1$
P
U
L
G
$G_1$
PU
G
L
1
HU
$L_1$
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G
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VS
PS
Z
G
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HS
1
$L_1$

Fig. 3.

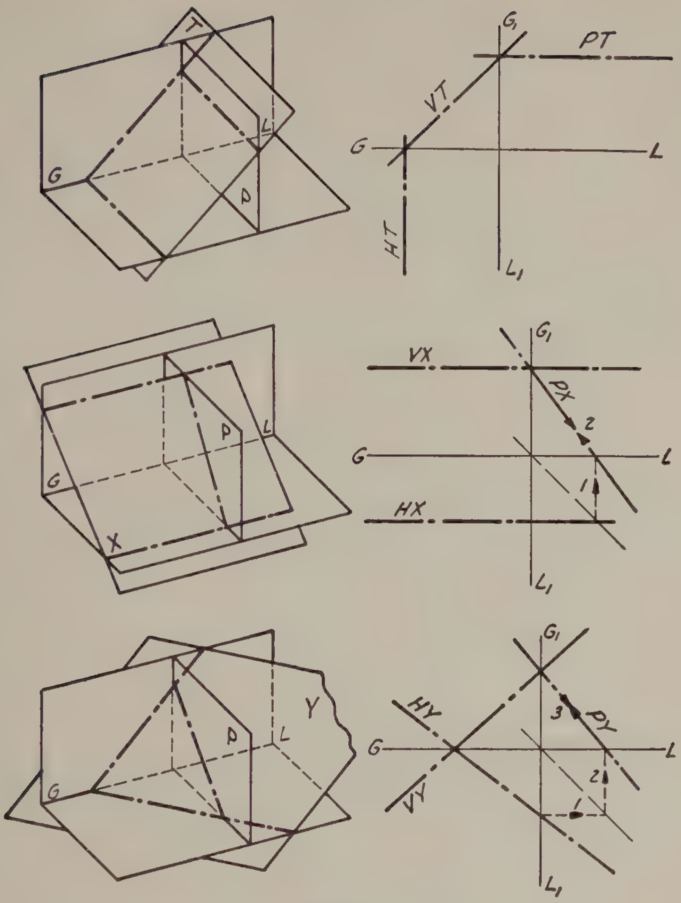


Fig. 3.

together with the characteristic directions of their traces, are as follows:

1. Parallel to *H*—*V* trace parallel to *GL*, *H* trace at infinity.
2. Parallel to *P*—Both traces perpendicular to *GL*.
3. Parallel to *V*—*H* trace parallel to *GL*, *V* trace at infinity.
4. Perpendicular to H { *V* trace perpendicular to *GL*.
   and inclined to *V* { *H* trace inclined to *GL*.
5. Perpendicular to *V* { *V* trace inclined to *GL*.
   and inclined to *H* { *H* trace perpendicular to *GL*.

6. Parallel to *GL* and inclined to *H* and *V*—Both traces parallel to *GL*.
7. Inclined to *H*, *V*, and *GL*—Both traces inclined to *GL*.

The utility of the traces as indicating readily the directional relation of planes so defined with the coordinate planes is easily appreciated. This was mentioned as one of the reasons for the use of these particular lines to define planes while we are dealing with the simpler relations of points, lines, and planes.

See Test Exercise 12, page 239.

## EXTENT OF PLANES

**39.** The planes shown in Fig. 3 are not limited to any quadrant but pass through two, three, or four quadrants, according to their position in any particular problem. Though the construction of a problem may deal with only that portion of a plane which lies in one quadrant, it must be understood that the plane is unlimited in extent. The traces therefore are not limited by *GL*. Figure 4(*a*) shows four ways of representing the same plane by its traces.

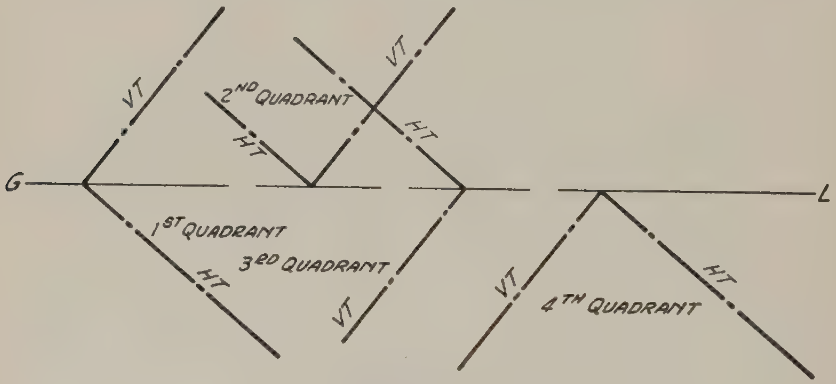


FIG. 4(*a*).

Since planes are defined by any pair of intersecting or parallel lines, it is to be expected that they will be defined by such a polygon as *ABCD*, Fig. 4(*b*). But since, to repeat, planes are unlimited in extent, no matter how defined or described, extensions of all the sides of the polygon and all lines connecting points on these sides lie in the plane of the polygon.

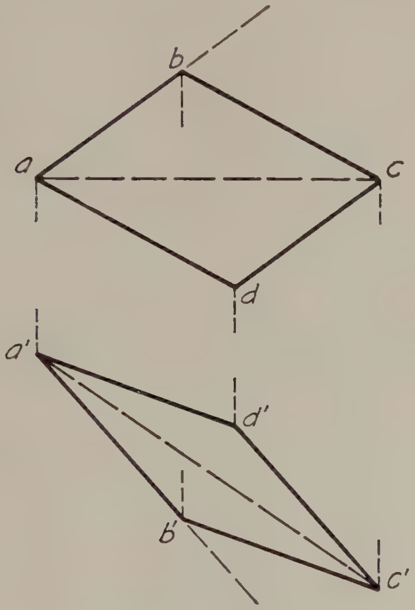


Fig. 4(*b*).

# CHAPTER IV

## SIMPLE RELATIONS OF THE POINT, LINE, AND PLANE

### TO ASSUME A PLANE

**40.** A plane whose position and direction are not specified may be assumed by describing any two intersecting or parallel lines. If desired or convenient, the two intersecting lines may be the intersections of the plane with the coordinate planes, that is, traces, which must either intersect on *GL* or be parallel to it, Fig. 1.

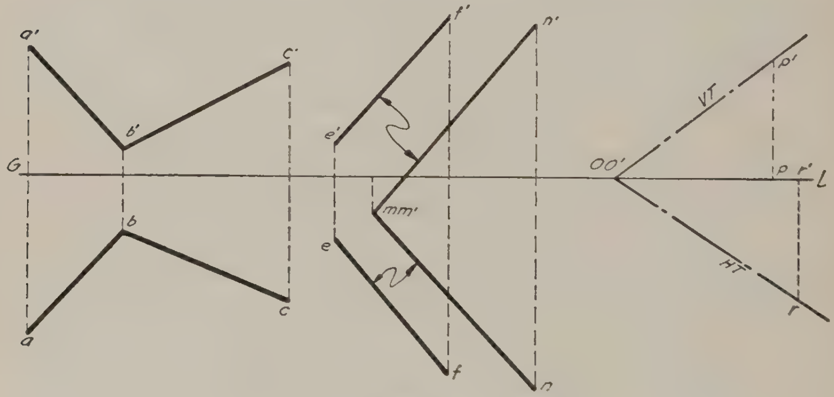


FIG. 1.

See Test Exercise 13, page 240.

**41.** Problem 6. **To assume a line in a given plane.**

**Analysis:**

A plane is defined by any two lines in it.

A line lies in a plane if it has two points in common with it.

A line through or connecting any two points on the two given lines must lie in the plane which they define.

A line intersecting one of the given lines and parallel to the other given line will lie in the plane which they define.

**Construction:**

Given: Planes $T$, Figs. 2($a$–$f$).

Required: Lines in the respective planes.

If the plane is defined by those special lines which we call *traces*, Fig. 2($a$), the points $V$ and $H$ (*1,2*) and (*3,4*) chosen anywhere on these lines are the respective piercing points of the assumed line in the coordinate planes. If the plane is defined by any two lines as in Fig. 2($d$), any two points, as $A$ and $C$, or $D$ and $E$, may be connected by lines (*1,4*) and (*5,6*).

A plane is defined by any three points of it. If then a polygon of any number of sides, e.g., Fig. 2($c$), has one projection given and the other projections of at least three corners, that projection may be completed by extending the two completely determined sides or by the use of auxiliary coplanar lines.

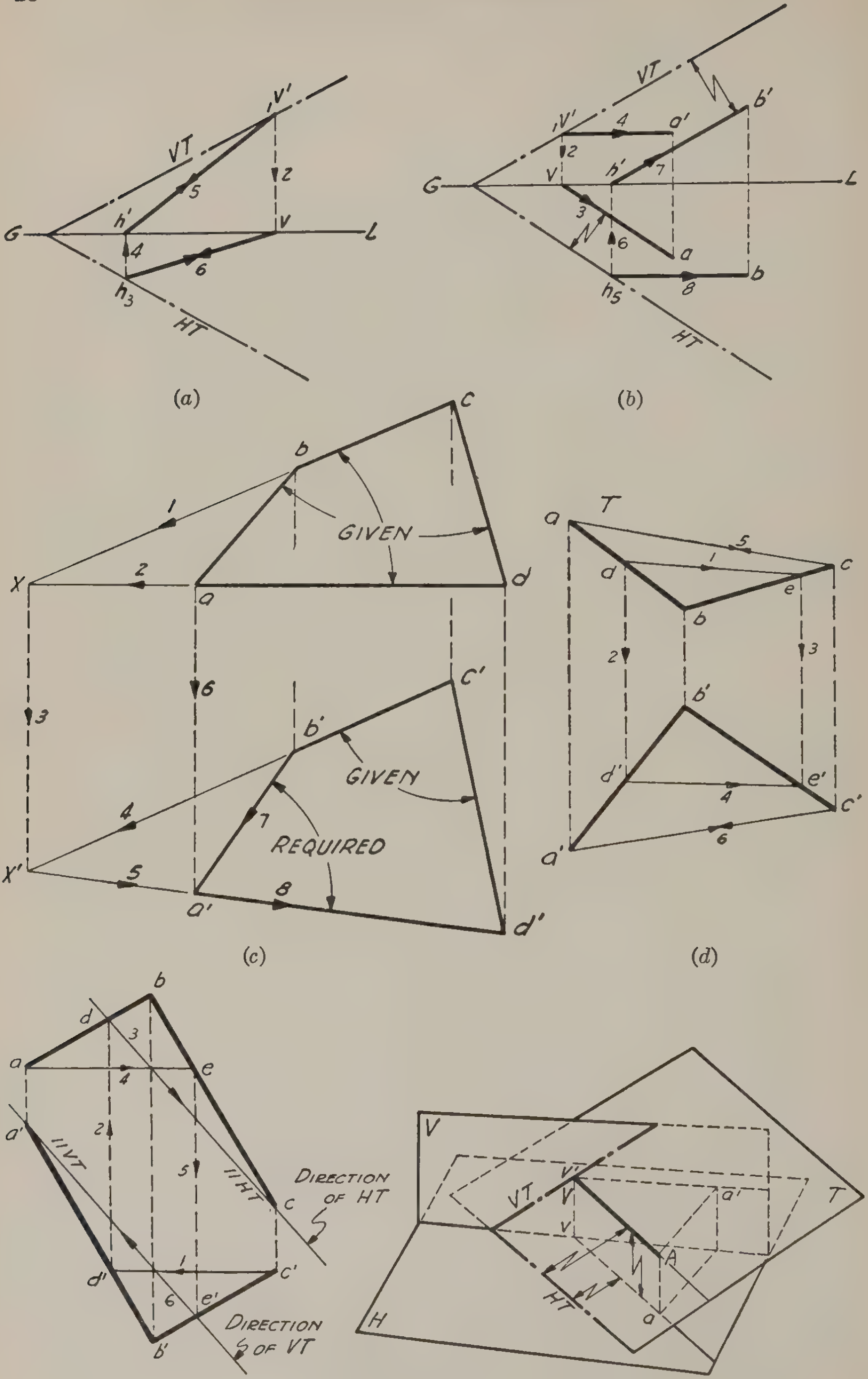
Any horizontal line in a given plane is called a **horizontal** of that plane, e.g., $VA$ of plane $T$, Fig. 2($f$). The $H$ projection of such a line is parallel to the $H$ trace of the plane, for the line $VA$ is parallel to $HT$ and to its own $H$ projection $va$. $HT$ and $va$ are therefore parallel to each other.

Any line which lies in a given plane and is parallel to the $V$ plane is called a **vertical** of that plane, and by proof similar to the above its $V$ projection is parallel to the $V$ trace, Fig. 2($b$).

Any line which lies in a given plane and is parallel to the $P$ plane is called a **profile** of that plane, and its $P$ projection is parallel to the $P$ trace.

The fact of great importance concerning *horizontals, verticals, and profiles* is that the *directions of the traces* of a plane may be determined when the plane is defined by any two lines in it merely *by assuming* in that plane *a horizontal line, a line parallel to V or a line parallel to P.* This is simple under any and all circumstances; for a horizontal line $CD$ (*1,3*), Fig. 2($e$) is always identified by its horizontal $V$ projection, $c'd'$. A line parallel to $V$, $AE$ (*4,6*), is identified by its horizontal $H$ projection, $ae$.

**42.** Problem 6: Example 1. **To assume a point in a given plane.**


VT
V'
2
5
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h'
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4
6
h3
HT
VT
b'
V'
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a'
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G
v
h'
L
3
6
a
h5
8
b
HT
c
b
1
GIVEN
X
2
a
d
6
3
c'
b'
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4
7
REQUIRED
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8
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d
1
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3
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Direction of HT
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b'
Direction of VT
V
VT
v'
v
a'
T
A
HT
a
H

(a) (b) (c) (d) (e) (f)

FIG. 2.

**Analysis:**

A plane is defined by any two lines in it.

A point is known to lie in a plane if it lies on a line of the plane.

A line lies in a plane if it has two points in common with it.

**Solution:**

Any point on either of the two lines which define a plane, or upon any line joining points of these two lines, lies in the given plane, Figs. 2(*a–f*).

43. Problem 6: Example 2. **To pass a plane through a given line.**

**Analysis** (*a*):

Any two intersecting or parallel lines define a plane.

**Solution** (*a*):

If through any point of the given line a second line is passed, or if a second line is assumed parallel to the given line, the planes so defined pass through the given line.

**Analysis** (*b*):

Since by Article 41 a line which lies in a plane $T$ pierces $V$ and $H$ in points which lie on the $V$ and $H$ traces of $T$, obviously the converse, that the traces of a plane pass through the respective piercing points of all lines in it, is true.

**Solution** (*b*):

If plane traces are passed through the respective piercing points of the given line, and intersecting each other on $GL$ or both parallel to it, they define a plane containing the given line.

44. Problem 7. **To determine the traces of a plane defined by any two intersecting or parallel lines.**

**Analysis** (*a*):

The traces of a plane are the loci of the corresponding piercing points of all lines of the plane.

**Solution** (*a*):

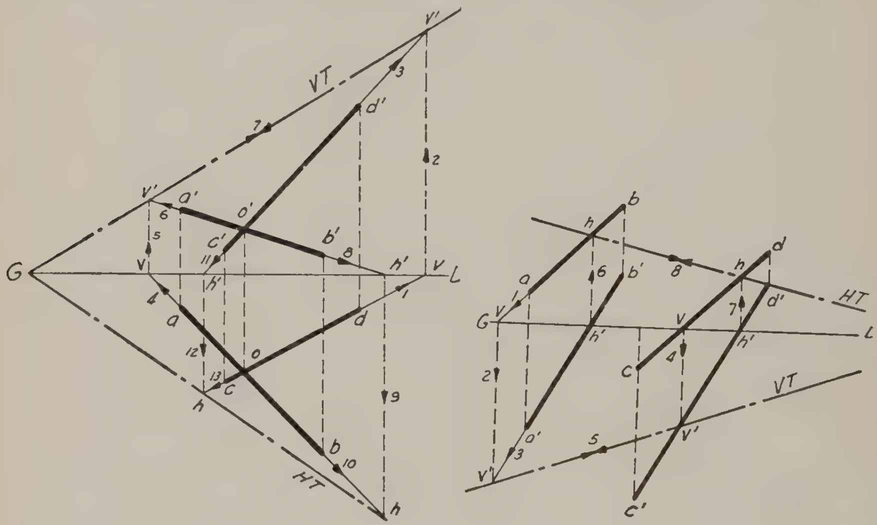
If the $V$, $H$, and $P$ piercing points of the given lines are determined, they will define the corresponding traces of the plane. The $V$ and $H$ traces will intersect on $GL$ or be parallel to it. The $V$ and $P$ traces will intersect on $G_1L_1$ or be parallel to it.

**Construction** (*a*):

Given: Lines *AB* and *CD*, Figs. 3(*a*) and (*b*).

Required: Traces of the plane of *AB* and *CD*.

By Article 30, the projections of the *V* piercing points of *AB* and *CD*, Fig. 3(*a*), are $v$, $v'$ (*1,3*) and $v$, $v'$ (*4,6*); and the projections of the *H* piercing points are $h$, $h'$ (*8,10*) and $h$, $h'$ (*11,13*). The *V* projection of the *V* trace of the required plane *T* passes through $v'$ and $v'$ (*7*); its *H* projection lies in *GL*, through $v$ and $v$. The *H* projection of the *H* trace passes through $h$ and $h$ (*14*); its *V* projection lies in *GL*, through $h'$ and $h'$. The traces must either intersect on *GL* or be parallel to *GL*.



(*a*) (*b*)

FIG. 3.

**Analysis** (*b*):

The *H* and *V* piercing points of all lines in a plane will be found on the corresponding traces of the plane.

If the piercing points of certain lines are inaccessible, other lines may be assumed [Article 41] in the plane and their piercing points used to determine the traces.

The directions of the traces of a plane may be determined by assuming a horizontal and a vertical of the plane. A trace may thus be determined by one of its points and its direction.

**Solution** (*b*):

If any convenient point of the one given line is connected with any point of the second given line, Fig. 4, the piercing points of this auxiliary line are points of the required traces of the plane.

**45.** Problem 7: Example 1. **To determine the traces of a plane defined by a given point and a given straight line.**

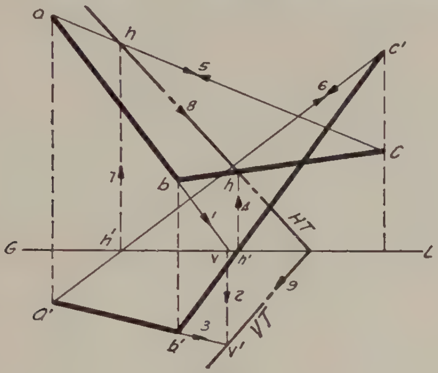


Fig. 4.

**Analysis:**

The plane of a given line and a given point, Fig. 5, is defined by the given line and a line connecting the given point with any point of the line.

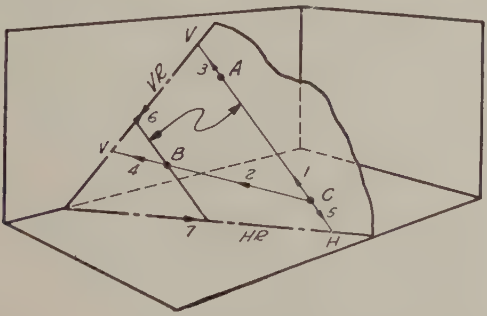


Fig. 5.

The plane is defined also by the given line and a line through the point parallel to the given line.

**Solution:**

The traces of the plane of the given point and line (as *AB* and *C*, Figs. 3(*b*) and 4) will pass through the respective piercing points of the given line and the auxiliary parallel or intersecting line.

46. Problem 7: Example 2. **To determine the traces of a plane defined by any three points.**

**Analysis:**

Three points define a plane because they define two or three intersecting lines joining them, Fig. 6.

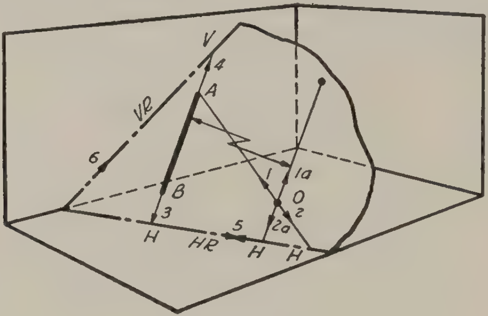


Fig. 6.

**Solution:**

The required traces pass through the respective piercing points of the lines joining the given points (as points *A*, *B*, and *C*, Figs. 3(*b*) and 4).

See Test Exercises 14 and 15, pages 241 and 242.

## LINES PARALLEL TO PLANES

**47.** The only requirement that a line need fill in order to be parallel to a plane is that it be parallel to any line of that plane. Thus *DE* is parallel to plane *ABC*, Fig. 7, because it is parallel to *BC*. If plane *R* is defined by its traces, which are its lines of intersection with *V* (*FM*) and *H* (*FN*), Fig. 8, and lines are desired through point *O* parallel to *R*, one line *OP* may be drawn

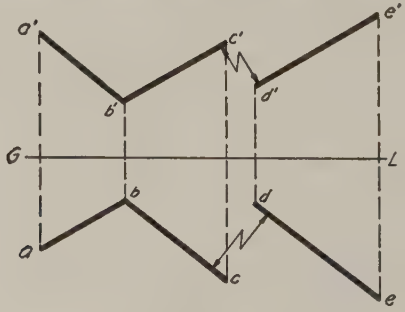


Fig. 7.

parallel to *VR* (or *FM*), (*1*,*2*), another parallel to *HR* (or *FN*), (*3*,*4*), and a third, *OS*, parallel to *MN* (*5*,*6*), which is a line of *R* since it passes through two points in *R*. Also, if planes *R* and *S* are parallel, every line of one plane is parallel to the other plane.

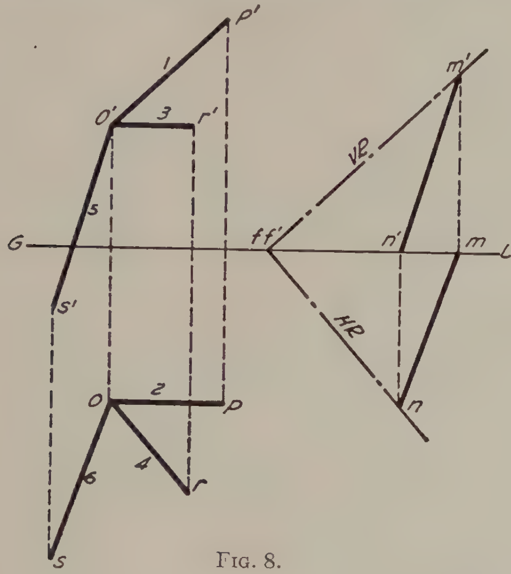
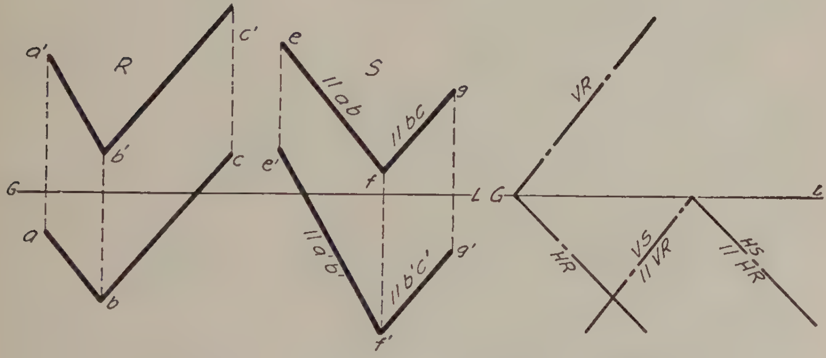


Fig. 8.



(*a*) (*b*)

Fig. 9.

## PARALLEL PLANES

**48.** If two intersecting lines of one plane, *R*, are respectively parallel to two intersecting lines of another plane, *S*, Figs. 9(*a*) and 9(*b*), the planes *R* and *S* are parallel; that is, they are everywhere equally distant from each other.

From the above it follows that, *if in one plane R two intersecting lines can be determined which are respectively parallel to two lines lying in plane S, the two planes are proved parallel.*

49. Problem 7: Example 3. **Through one given line to pass a plane parallel to a second given line.**

**Analysis:**

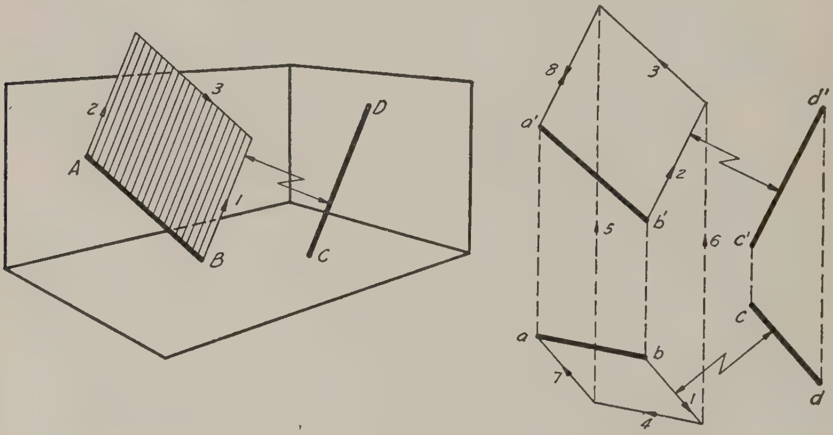
The given lines are assumed to be non-parallel and non-intersecting.

A plane is defined by two intersecting or parallel lines.

A plane is parallel to a given line if it contains a line parallel to that given line and does not contain the given line.

**Solution:**

If through any point of the first given line, an auxiliary line parallel to the second given line is passed, the plane defined by these two intersecting lines is the required plane, Figs. 10(*a*) and 10(*b*).



(*a*) (*b*)

Fig. 10.

50. Problem 7: Example 4. **Through a given point to pass a plane parallel to a given line.**

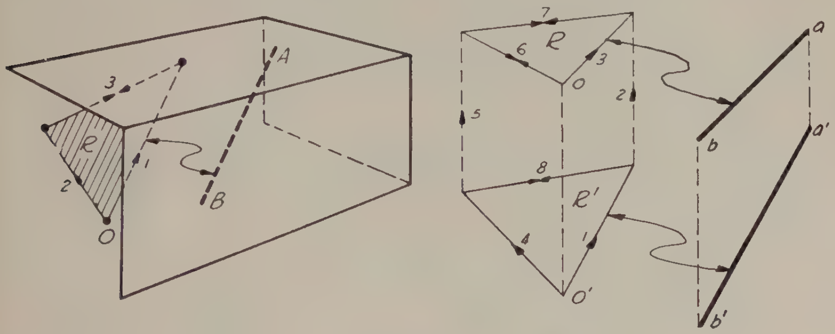
**Analysis:**

A plane passes through a given point if the point lies on a line of that plane.

A plane is parallel to a given line if it contains a line parallel to the given line.

**Solution:**

If through the given point a line is passed parallel to the given line, any plane containing this line will satisfy the required conditions, Figs. 11(*a*) and 11(*b*), provided it does not contain the given line.



(*a*) (*b*)

FIG. 11.

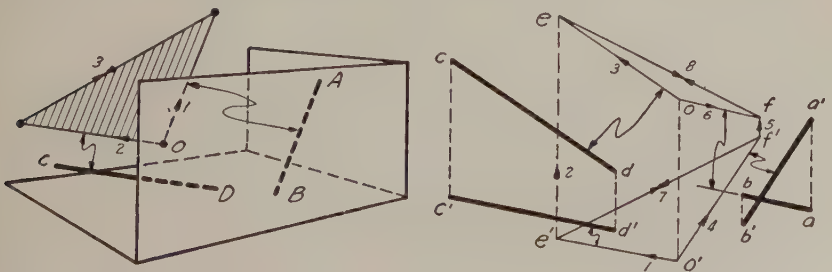
**51.** Problem 7: Example 5. **Through a given point to pass a plane parallel to two given lines.**

**Analysis:**

A plane is parallel to a given line if it contains a line parallel to the given line.

**Solution:**

If through the given point two lines parallel respectively to the given lines are passed, the plane defined by these intersecting lines is the required plane, Figs. 12(*a*) and 12(*b*).



(*a*)] (*b*)

FIG. 12.

**52.** Problem 7: Example 6. **Through a given point to pass a plane parallel to a given plane.**

**Analysis:**

A plane passes through a point if the point lies on a line of the plane.

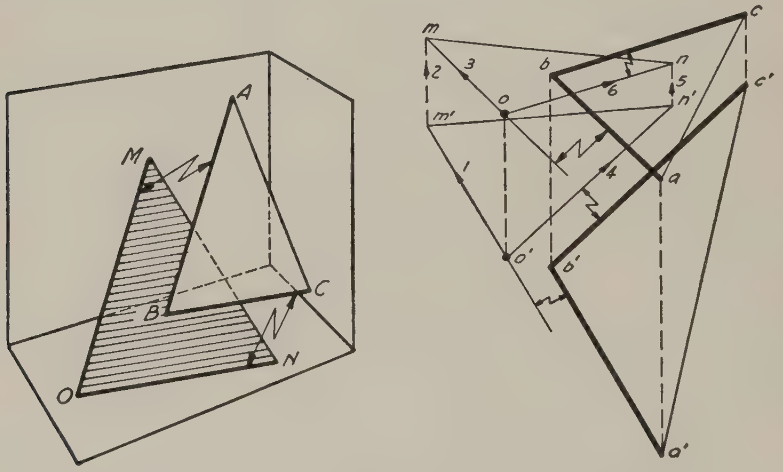
Two intersecting lines define a plane.

If two intersecting lines of one plane are parallel respectively to two intersecting lines of another plane, the planes are parallel.

The corresponding traces of parallel planes are parallel.

**Solution** (*a*):

If two lines are passed through the given point parallel to two intersecting lines of the given plane, Fig. 13(*a*), they will define the required plane.



(*a*) (*b*)

Fig. 13.

**Construction** (*a*):

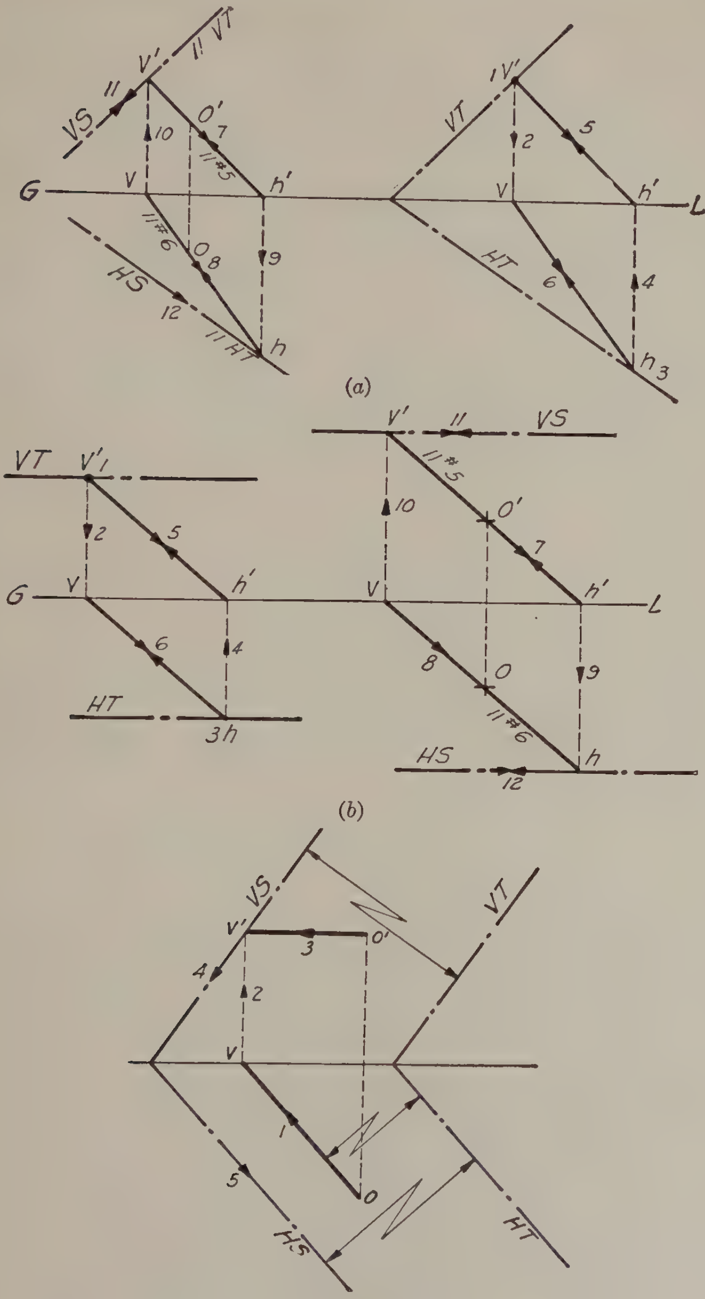
Given: Plane *ABC* and point *O*, Fig. 13(*b*).

Required: A plane through point *O* parallel to plane *ABC*.

Through point *O* lines *OM* (*1,3*) and *ON* (*4,6*) are passed parallel respectively to lines *AB* and *BC*. Plane *MON* is the required plane.

**Solution** (*b*), (Traces):

If through the given point a line is passed parallel to a line of the given plane, the required traces will pass through the respective piercing points of this line and will be parallel to the corresponding traces of the given plane.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11#6
HS
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12
VS
VT
V'
3
O'
4
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V
1
5
O
HS
HT

(a)

(b)

(c)

FIG. 14.

**Construction** (*b*):

Given: Plane *T* and point *O*, Figs. 14(*a–c*).

Required: The traces of plane *S* through point *O* parallel to plane *T*.

Any line *VH*, *vh*, *v'h'* (*1,6*) is assumed in plane *T*. A line, *VH*, is passed through point *O* parallel to line *VH* of plane *T* (*7,10*). The traces of plane *S* pass through *v'* and *h* (*11,12*) parallel to the corresponding traces of plane *T*, Figs. 14(*a*) and 14(*b*).

In many cases the construction may be simplified by passing the auxiliary line through point *O* and parallel to one of the lines given in plane *T*, i.e., parallel to one of its traces, Fig. 14(*c*).

See Test Exercises 16, 17, 18, and 19, pages 243–246.

## PRINCIPLE "B"

## PERPENDICULARITY

**53.** Lines in or parallel to any given plane and parallel respectively to the coordinate planes of projection (termed **"horizontals"** and **"verticals"**) serve to *define that plane for purposes of perpendicularity.*

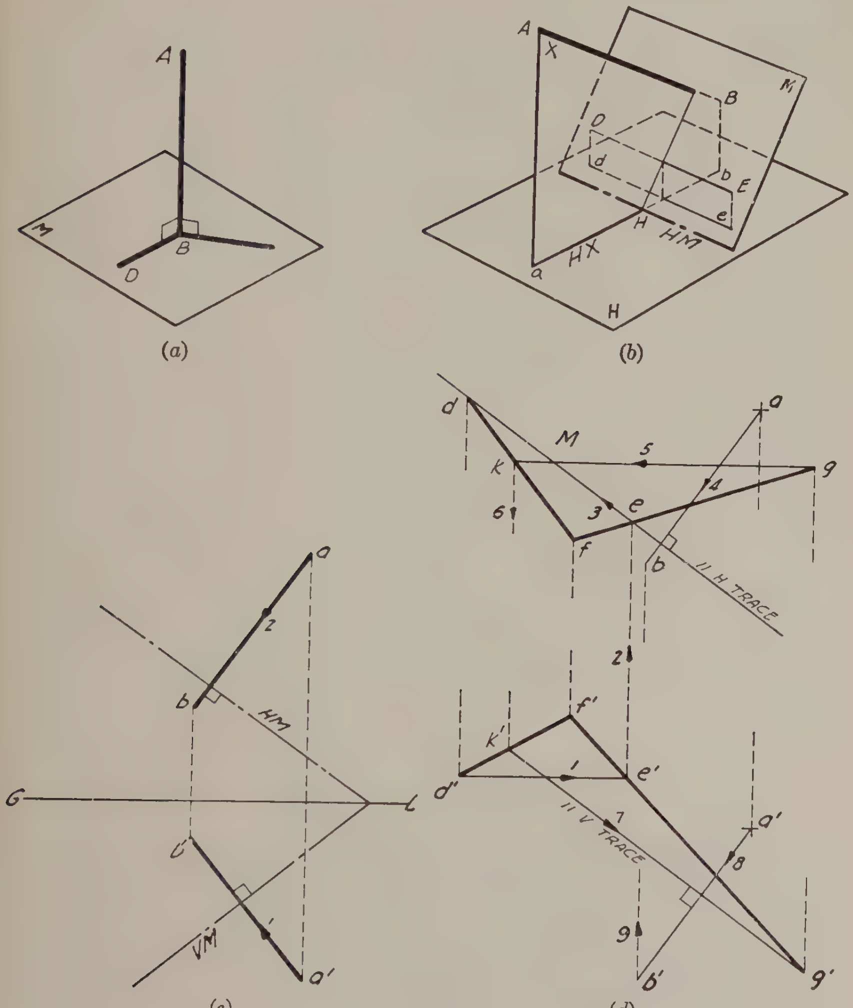
## LINES PERPENDICULAR TO PLANES

**54.** A line, *AB*, Fig. 15(*a*) is perpendicular to a plane if it is perpendicular to two lines, such as *BC* and *BD*, in that plane. For our purposes in descriptive geometry it is necessary to add to the above statement of the condition of perpendicularity the circumstance under which we may test and pass upon the right-angle relation between lines. As yet we may recognize the relation of perpendicularity between two lines from their descriptions or projections *only when at least one of the two lines is parallel to a coordinate plane.* In that case the angle of ninety degrees is projected in its true value on the corresponding coordinate plane [Article 33]. For the present then we are limited to such combinations to construct a perpendicular to a plane or to test the condition of perpendicularity.

**Proof:**

Given: Planes *H* and *M*, and line *AB* perpendicular to plane *M*, Fig. 15(*b*).

Required: Evidence in the relations of the projections of



(*a*) (*b*) (*c*) (*d*)

Fig. 15.

*AB* to projections of lines in *M* of its condition of perpendicularity to plane *M*.

The given planes *M* and *H* intersect in the line *HM*, which is the trace of plane $M$ in plane $H$. Line $AB$ is given perpendicular to plane $M$. Plane $X$ through $AB$ is the $H$ projecting plane of $AB$, hence is perpendicular to $H$. The $H$ trace of plane $X$ is $HX$, which coincides with $ab$, the $H$ projection of the line $AB$.

Plane $X$ is perpendicular to plane $M$ since it passes through the line $AB$, which, by hypothesis, is perpendicular to the plane. Plane $X$ is constructed perpendicular to plane $H$. It is therefore perpendicular to the line of intersection $HM$ of these two planes. Conversely, $HM$ is perpendicular to plane $X$. It is therefore perpendicular to every line in plane $X$ passing through its piercing point $H$ in $X$; $ab$, the $H$ projection of the line $AB$, is one such line. The conclusion then is that: *the projections of a line which is perpendicular to a plane are perpendicular to the corresponding traces of that plane*, Fig. 15($c$).

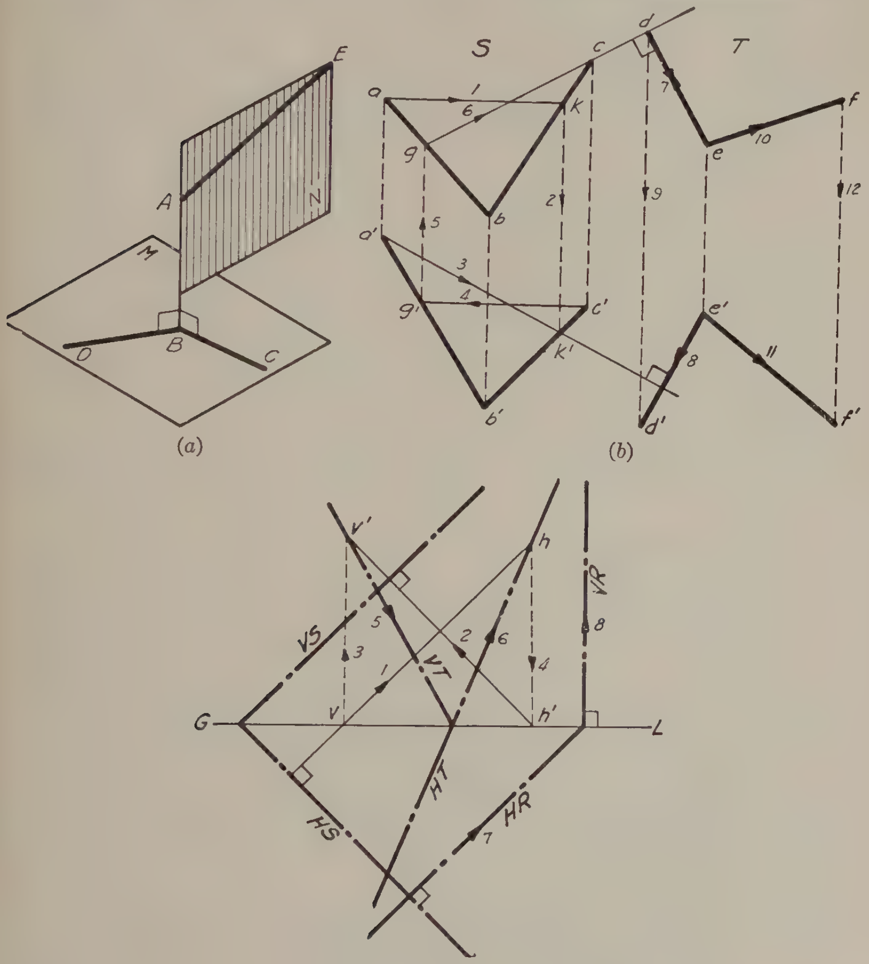
Line $DE$ in plane $M$ is drawn parallel to $H$, that is, horizontal. It is parallel to every horizontal line in plane $M$, hence to $HM$. $DE$ is parallel also to its own $H$ projection $de$. $HM$ and $de$ are thus parallel to $DE$, and therefore to each other. The $H$ projection $ab$ of line $AB$ is hence perpendicular to $de$ as well as to $HM$ and therefore to the $H$ projections of all horizontal lines in plane $M$. The additional important conclusion then is that: If a line is perpendicular to a plane, its $H$, $V$, and $P$ projections are perpendicular respectively to the corresponding projections of any horizontal, vertical, and profile of the plane, Fig. 15($d$).

## PERPENDICULAR PLANES

**55.** Two planes are perpendicular if one of them has been constructed to contain, or can be shown to contain, a line which is perpendicular to the other plane, Fig. 16($a$). Such a line can be constructed or can be tested by the method of Article 54. Such a line and any other line intersecting it will define a plane which is perpendicular to the given plane.

If two planes are described by pairs of intersecting or parallel lines other than the traces, there is no means of determining the angular relation between the planes from mere examination of the projections of such lines.

Planes *R* and *S*, Fig. 16(*b*), were constructed perpendicular to each other by making line *DE* in plane *T* perpendicular to plane *S*. However, if the projections of the *vertical* (*1,3*) and *horizontal* (*4,6*) of plane *S* were omitted, the perpendicularity of the planes would not be evident from inspection of the projections of lines *ABC* and *DEF*.



(*a*) (*b*)

(c)

Fig. 16.

If planes are described by their traces, the condition of perpendicularity cannot be determined by inspection unless one of the two planes is also perpendicular to a plane of projection. In such case, Fig. 16(*c*), the traces in that coordinate plane are perpendicular. Planes *S* and *T* are perpendicular since plane *T* is shown to contain a line perpendicular to plane *S*.

**56.** Problem 7: Example 7. **To pass a plane through a given point perpendicular to a given line.**

**Analysis:**

A plane which is perpendicular to a given line intersects that line in a point.

The given line is perpendicular to each line in the above plane passing through the point of intersection.

The condition of perpendicularity can be *seen* between the given line and only three of the lines of the plane. These are a horizontal line (a *horizontal*), a line parallel to *V* (a *vertical*), and one parallel to *P* (a *profile*), (Principle of Perpendicularity [Article 53]).

A plane through a given point and parallel to any plane which is perpendicular to a given line is also perpendicular to that line.

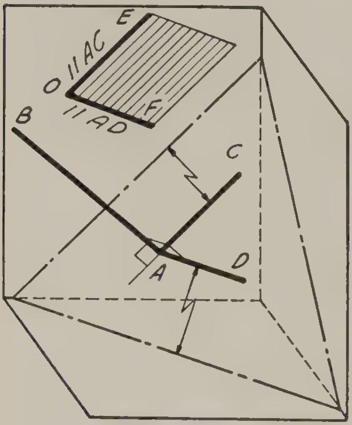


FIG. 17(*a*).

**Solution:**

If through any point of the given line, Fig. 17(*a*), two lines are drawn perpendicular to the line [Article 33], they constitute a plane perpendicular to the line. A plane through the given point parallel to this plane is the required plane. The traces of this plane, if they are required, are perpendicular to the corresponding projections of the given line.

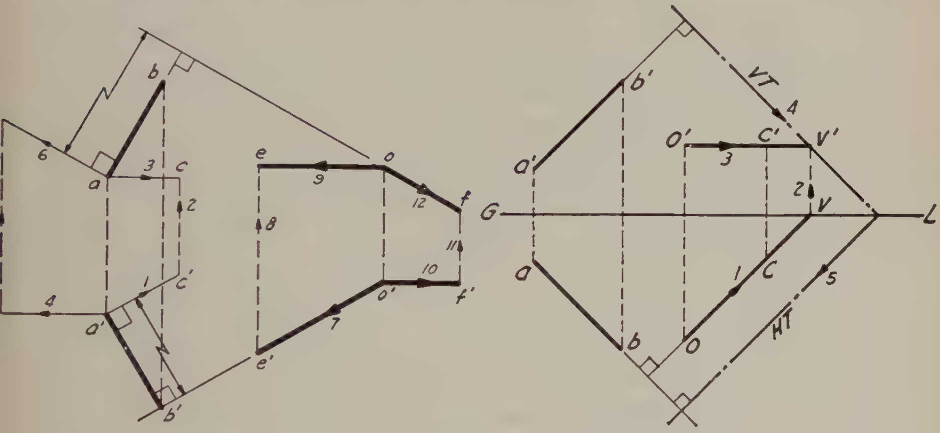
**Note:** This problem becomes available to pass a line through a given point perpendicular to a given line after completing Principle "C" (intersection of any line with any plane).

**Construction:**

Given: Line *AB* and point *O*, Figs. 17(*b*) and 17(*c*).

Required: A plane through point *O* perpendicular to line *AB*.

Through point *A* two lines are passed perpendicular to *AB*, Fig. 17(*b*). Line *AC* (*1,3*) is parallel to *V*, and the angle of 90 degrees therefore shows on the *V* plane. Line *AD* (*4,6*) is horizontal, and the angle of 90 degrees shows on the *H* plane. Plane *DAC* is perpendicular to *AB*. Lines *OE* and *OF* are passed through *O* parallel, respectively, to *AC* (*7,9*) and *AD* (*10,12*). Plane *EOF* is parallel to plane *DAC* and hence is perpendicular to *AB*. It is apparent that lines *OE* and *OF* could have been passed through *O* without passing lines *AD* and *AC* through *A* at all, because *OE* is a *vertical* of the required plane and *e′o′* (*7*) is hence perpendicular to *a′b′*; and *OF* is a *horizontal* of the required plane, hence *fo* (*12*) is perpendicular to *ab*.



(*b*) (*c*)

FIG. 17.

In Fig. 17(*c*) a *horizontal*, *OC*, of the required plane is passed through point *O* (*1,3*). The *V* trace of plane *T* then passes through *v′* perpendicular to *a′b′* (*4*). *HT* is perpendicular to *ab* (*5*).

**57.** Problem 7: Example 8. **To pass a plane through a given point perpendicular to a given plane.**

**Analysis:**

A plane passes through a point if the point lies on a line of the plane.

A plane, *S*, is perpendicular to a second plane, *T*, if *S* contains a line which is perpendicular to *T*.

A line is perpendicular to a plane if it is perpendicular to two intersecting lines of that plane.

A line is known to be perpendicular to a plane if its respective projections are perpendicular to the *directions* of the corresponding traces of the plane.

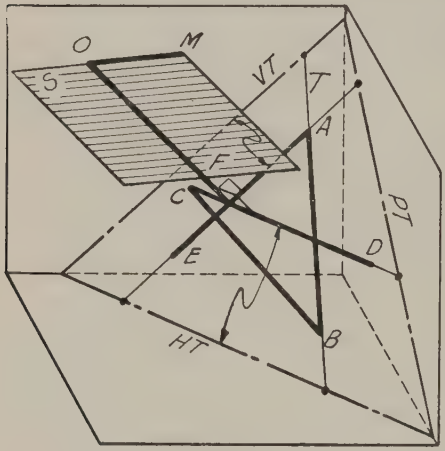


Fig. 18(*a*).

The directions of the traces of a plane are known from the directions of the *H* projection of a *horizontal,* the *V* projection of a *vertical,* and the *P* projection of a *profile.*

**Solution:**

If through the given point *O* a line *OF* is passed perpendicular to the given plane, Fig. 18(*a*), any plane passing through line *OF* will be perpendicular to the given plane.

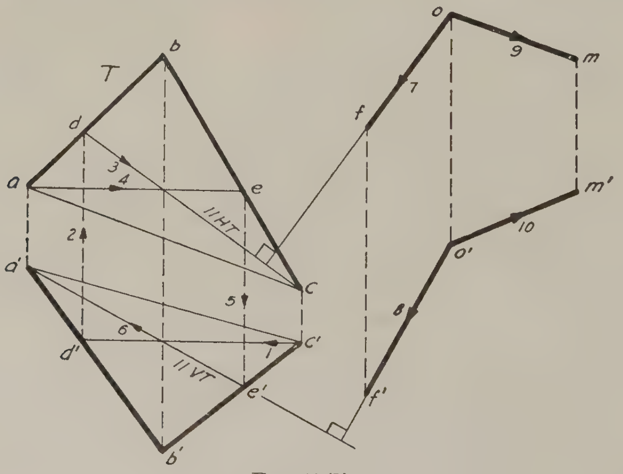


Fig. 18(*b*).

**Construction:**

Given: Point *O* and plane *T* (*ABC*), Fig. 18(*b*).

Required: A plane through point *O* perpendicular to plane *T*.

*CD* (*1,3*) is a *horizontal* of plane *ABC*. *AE* (*4,6*) is a *vertical* of plane *ABC*. The *H* projection *dc* (*3*) of *DC* and the *V* projection *a'e'* (*6*) of *AE* are the *directions*, respectively, of the *H* and *V* traces of *ABC*. The projections of a line through *O* perpendicular to *ABC* are hence *fo* and *f'o'* (*7,8*) perpendicular, respectively, to *dc* and *a'e'*. Any other line *OM* (*9,10*) through *O* then defines with *OF* a plane *FOM* through *O* perpendicular to *ABC*.

58. Problem 7: Example 9. **To pass a plane through a given line, perpendicular to a given plane.**

### Analysis:

A plane, *S*, is perpendicular to a second plane, *T*, if *S* contains a line which is perpendicular to plane *T*.

If a line perpendicular to plane *S* is passed through any point of the given line, these two intersecting lines define the required plane.

See Analysis of Problem 7, Example 8.

### Solution:

If through any point of the given line a line is passed perpendicular to the given plane, the plane of this perpendicular and the given line will be the required plane.

### Construction:

Given: Line *OM* and plane *ABC*, Fig. 18(*b*).

Required: A plane through *OM* perpendicular to *ABC*.

Line *OF* is passed through *O* perpendicular to *ABC* as explained under the "Construction" in Article 57. Plane *FOM* is the required plane through *OM* and perpendicular to *ABC*.

59. Problem 7: Example 10. **To pass a plane through a given point perpendicular to two given planes.**

### Analysis (*a*):

A plane, *R*, is perpendicular to a plane, *S*, if *R* contains a line which is perpendicular to *S*.

Plane *R* is perpendicular to two planes *S* and *T* if it contains a line perpendicular to each.

**Solution** (*a*):

If through the given point *O*, Fig. 19(*a*), two lines are passed perpendicular, respectively, to the given planes, these two lines, *OA* and *OC*, define the required plane.

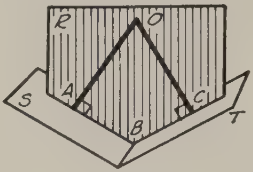


Fig. 19(*a*).

**Construction** (*a*):

Given: Planes *S* (*GKM*) and *T* (*DEF*), Fig. 19(*b*).

Required: A plane, *R*, through point *O* perpendicular to planes *S* and *T*.

The *horizontal NM* (*1,3*) and *vertical GW* (*4,6*) of plane *S* provide the directions of the *H* and *V* traces of plane *S*. The *horizontal DX* (*7,9*) and the *vertical FZ* (*10,12*) provide the directions of the traces of plane *T*. The projections of the perpendiculars *OA* and *OC* from point *O* to planes *S* and *T* are then drawn perpendicular to the respective traces of these two planes, *OA* (*13,15*) and *OC* (*16,18*). The plane of *OA* and *OC* is the required plane *R*.

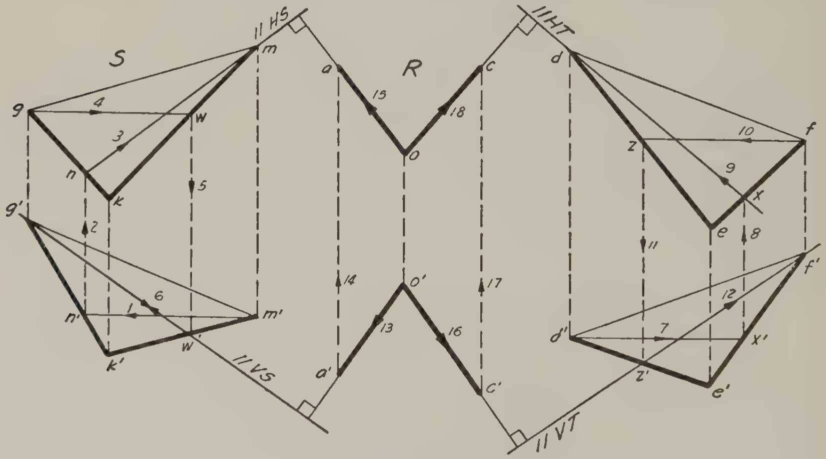


Fig. 19(*b*).

**Analysis** (*b*):

A plane, $R$, is perpendicular to a plane, $S$, if $R$ is perpendicular to a line in $S$. $R$ is hence perpendicular to two planes, $S$ and $T$, if it is perpendicular to lines in each, i.e., to their line of intersection.

**Solution** (*b*):

If through the given point $O$, Fig. 19(*a*), a plane is passed perpendicular to the line of intersection of the two given planes, this plane $OABC$ will be perpendicular to both given planes. (This method becomes available after completing Problem 10 [Article 64].)

See Test Exercises 20, 21, 22, and 23, pages 247–250

## PRINCIPLE "C"

## INTERSECTION

**60.** All points of one of two perpendicular planes are projected orthogonally upon the other plane in their line of intersection. Planes $X$ and $H$, Fig. 20(*a*), are perpendicular to each other. Plane $M$ intersects plane $X$ in line $P_1P_2$, which projects upon $H$ in $p_1p_2$ on $ab$. Line $AB$ of plane $X$ pierces plane $M$ in point $P_3$, which is projected upon $H$ in $p_3$ on $ab$.

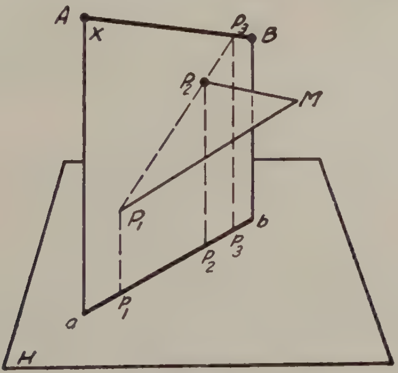


FIG. 20(*a*).

**61.** Problem 8. **To find the point in which a given line pierces the $H$ (or $V$) projecting plane of another line.**

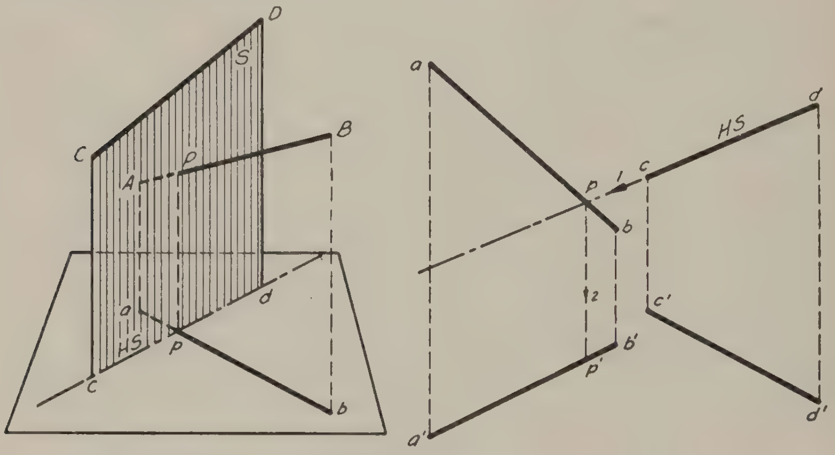
**Analysis:**

The *H* projection of a line is the locus of the *H* projections of all points of its *H* projecting plane.

The *H* projection of the desired point is hence common to the *H* projections of the two lines.

**Solution:**

The intersection of the *H* projections of the given lines, Fig. 20(*b*), is the *H* projection of the point in which either line pierces the *H* projecting plane of the other.



(*b*) (*c*)

FIG. 20.

**Construction:**

Given: Lines *AB* and *CD*, Fig. 20(*c*).

Required: The point *P* in which *AB* pierces the *H* projecting plane of *CD*.

The *H* trace of the *H* projecting plane of *CD* coincides with *cd* (*1*). The *H* projections of all points of plane *S* are found on the *H* trace, *HS*. The *H* projection of the point *P* in which *AB* pierces *S* is hence at *p* and the *V* projection at *p*′ (*2*).

62. Problem 9. **To find the point in which a given right line pierces a given plane.**

**Analysis:**

A point is always derived as the intersection of two lines.

The required point is common to the given line and the plane.

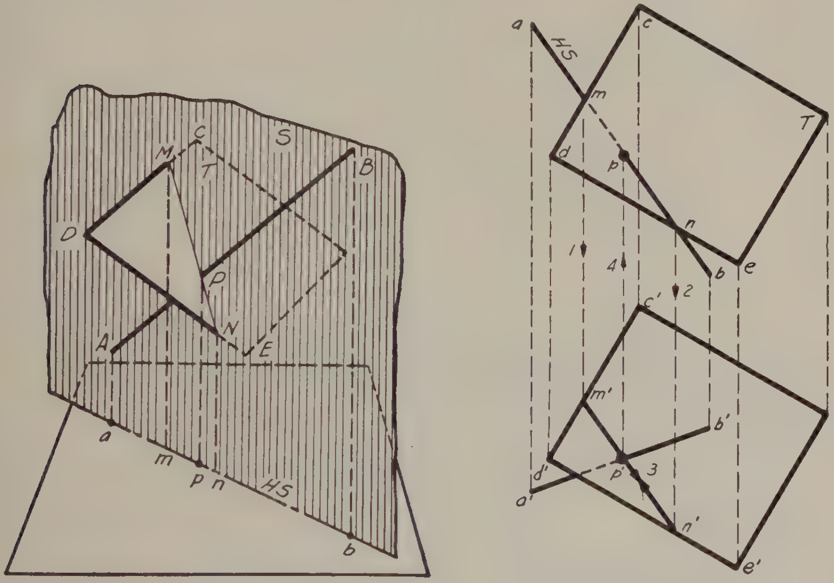
The auxiliary line in this case must lie in the given plane.

The auxiliary line and the given line define a plane which for convenience may be a projecting plane of the given line.

The auxiliary line passes through the points in which any two lines of the given plane pierce a projecting plane of the given line [Article 61].

**Solution:**

The required piercing point is the point of intersection of the given line and the line of intersection of the given plane with the *H* (*V* or *P*) projecting plane of the given line, Fig. 21(*a*).



(*a*) (*b*)

Fig. 21.

**Construction:**

Given: Line *AB* and plane *T* (*CDE*), Fig. 21(*b*).

Required: Piercing point of line *AB* in plane *T*.

The *H* projecting plane, *S*, is passed through *AB*. Its *H* trace coincides with *ab*. *CD* and *DE* pierce *S* in points *M* and *N* (*1*,*2*). The projections of the line of intersection *MN* of *S* and *CDE* are therefore *m*′*n*′ and *mn* (*1*,*3*). The *V* projection of *AB* intersects *m*′*n*′ at *p*′, the *V* projection of the required piercing point. The *H* projection of the piercing point, *P*, is at *p* (*4*) on *ab* and *mn*.

In Fig. 21(*c*) the intersection of the projecting plane *S* and the given plane *T* is found as in Solution (*b*), Problem 10 [Article 64].

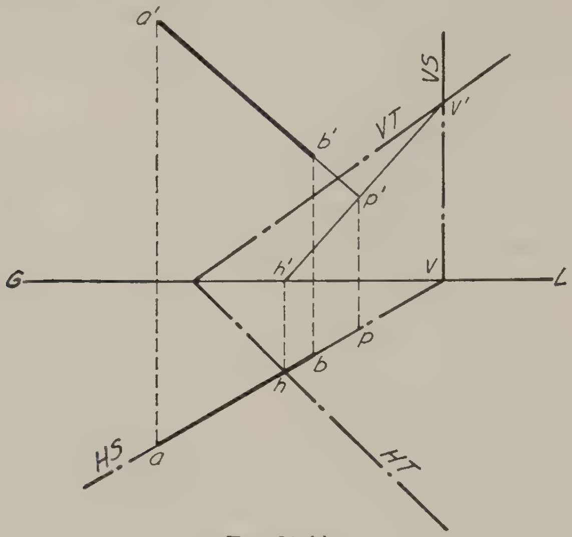


FIG. 21 (*c*).

## 63. Problem 9: Example 1. **To pass a line through a given external point and perpendicular to a given line.**

**Analysis:**

The locus of all lines perpendicular to a given line at any particular point of it is the plane through that point perpendicular to the line.

A plane through the given external point and perpendicular to the given line must contain the required perpendicular line.

The line joining the given point with the point in which the given line pierces the perpendicular plane is the required line.

**Solution:**

If a plane is passed through the given point perpendicular to the given line, the line joining the given point with the point in which the given line pierces this plane is the required line.

**Construction:**

Given: Point and line *AB*, Fig. 22.

Required: A line through *O* perpendicular to *AB*.

A plane is passed through point *O* perpendicular to *AB*. It is defined by the horizontal *OC* and the vertical *OD* (*1,6*). The piercing point *M* of *AB* in plane *COD* is determined by passing a vertical plane *X* through *AB* and finding the points *E* and *F*

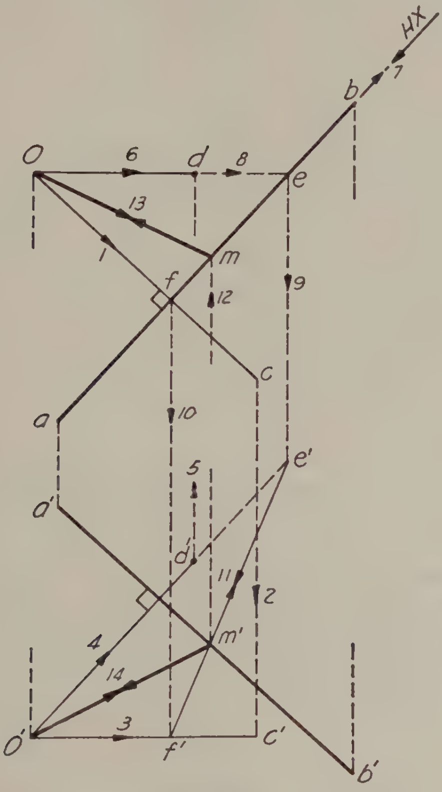


Fig. 22.

in which *OD* and *OC* pierce plane *X* (*7,12*). The required perpendicular is then *OM* (*13,14*).

64. Problem 10. **To find the line of intersection of any two given intersecting planes.**

**Analysis:**

Planes are defined by any two intersecting or parallel lines.

The intersection of two planes is a straight line defined by two points which are common to both planes.

The point in which a line pierces a plane is the point in which it intersects a line in that plane.

**Solution** (*a*):

If an auxiliary vertical plane, *S*, is passed through a line of one plane, Fig. 23(*a*), or wherever convenient, Fig. 23(*b*), it will intersect the given planes in lines which will intersect each other in a point of the required line of intersection. A second similar auxiliary plane will provide a second point.

**Construction** (*a*):

Given: Planes *ABCD* and *EFGH*, Figs. 23(*c*) and (*d*).

Required: The line of intersection.

Vertical plane *S*, Fig. 23(*c*), is passed through one of the given lines of plane *ABCD*, e.g., *DC* (*1*). Plane *ABCD* then intersects plane *S* in line *DC* (*2,4*) and plane *EFGH* intersects plane *S* in line *OR* (*5,7*). Since lines *DC* and *OR* both lie in plane *S*, they will intersect if they are not parallel. They intersect at $P_1$, since $d'c'$ intersects $o'r'$ at $p_1'$; $p_1$ is found on both *dc* and *or* (*8*).

A second vertical plane, *T* (*9*), is passed parallel to *S*, or in any convenient location. Plane *T* intersects plane *ABCD* in line *JK* (*10,12*) and plane *EFGH* in line *MN* (13,15). *JK* intersects *MN* in point $P_2$ (*16*). Points $P_1$ and $P_2$, which lie in both given planes, determine the required line of intersection, $P_1P_2$ (*17,18*). The process of determining the line of intersection of two planes defined by figures which do not overlap, Fig. 23(*d*), is the same as that just described for Fig. 23(*c*).

**Solution** (*b*):

If the two planes are defined by their traces, the *H* traces will intersect (unless parallel) in one of the required points and the *V* (or *P*) traces in another.

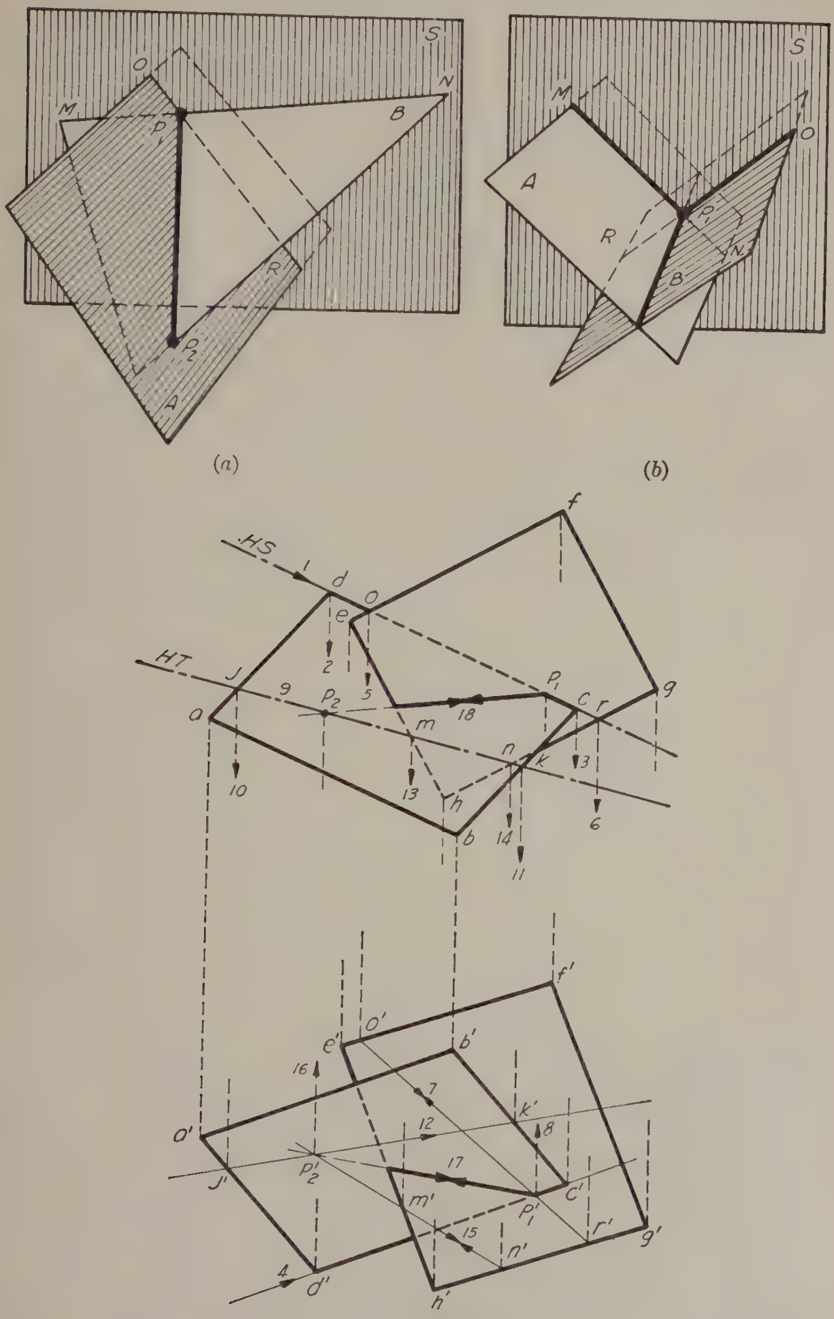
**Construction** (*b*):

Given: Planes *S* and *T*, Fig. 23(*e*).

Required: Line of intersection of *S* and *T*.

In this case the coordinate planes which are already provided constitute the auxiliary planes employed to intersect the given planes in lines which will intersect in points of the required line of intersection.

The intersection of the *V* traces of *S* and *T* is a point common to *S* and *T*. Since it lies also in the *V* plane, it is the *V*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HT
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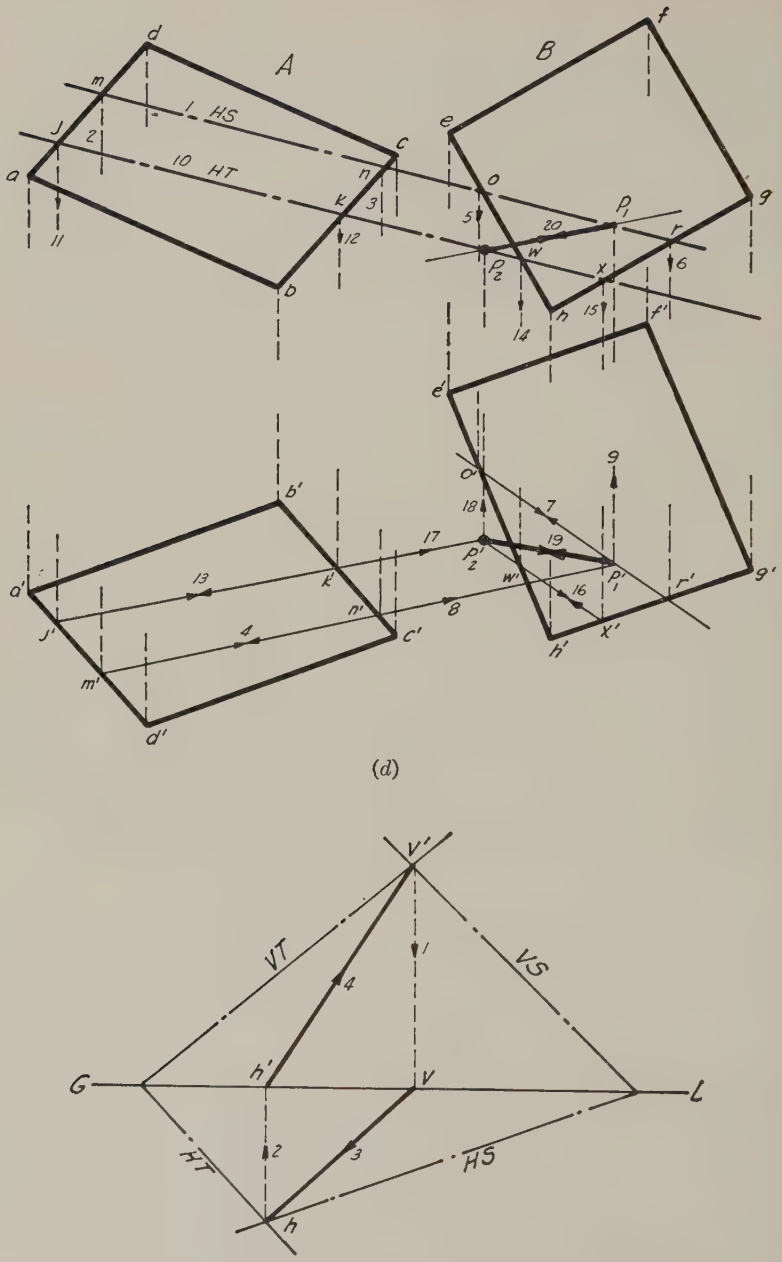
(a) (b)

(c)

Fig. 23.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(d)

(e)

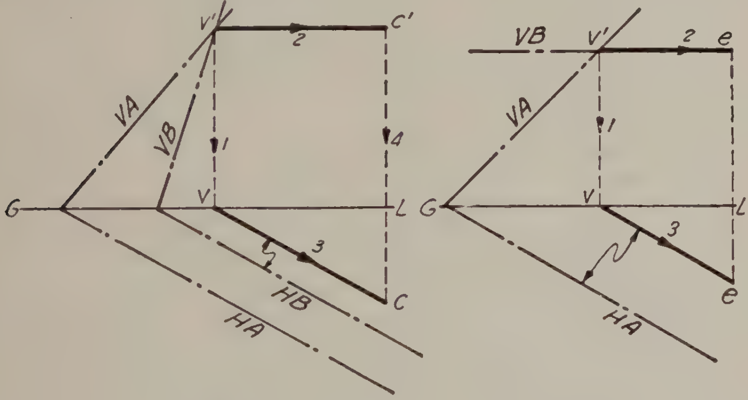
Fig. 23.

piercing point of the line of intersection; the *V* projection, *v′*, coincides with the point itself and, since it is a point in *V*, its *H* projection, *v*, is on *GL* (*1*). The *H* projection, *h*, of the *H* piercing point of the required line is the intersection of *HS* and *HT*; its *V* projection, *h′* (*2*), is on *GL*; *vh* and *v′h′* (*3*,*4*) are the *H* and *V* projections of the required line.

### SPECIAL CASES OF INTERSECTION OF PLANES

**65.** 1. If two corresponding traces are parallel, they may be considered as intersecting at infinity. The line of intersection then meets these parallel traces at infinity; i.e., it is parallel to them, Fig. 24(*a*).

2. If one plane is parallel to a coordinate plane, e.g., to *H*, the line of intersection is parallel to *H*; i.e., it is a *horizontal* of both planes. It is therefore parallel to the *H* trace of the oblique plane [Article 41], Fig. 24(*b*).



(*a*) (*b*)

Fig. 24.

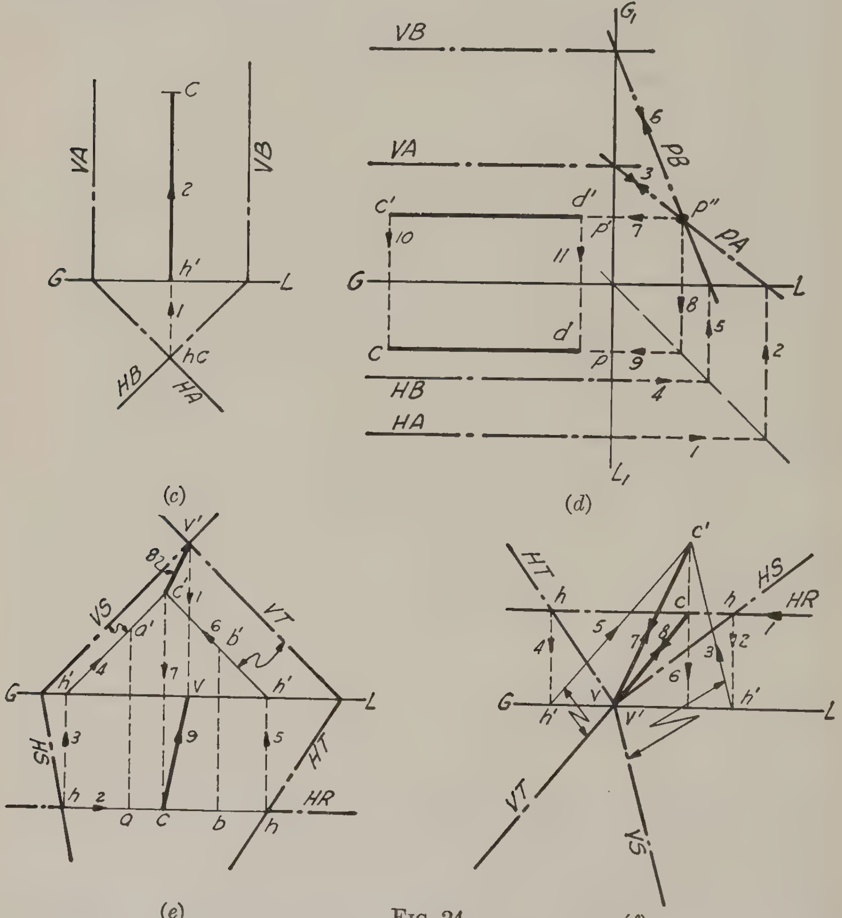
3. If both planes are perpendicular to *H* or *V*, the line of intersection is perpendicular to *H* or *V* [Article 25], (*1* or *2*), Fig. 24(*c*).

4. If both planes are parallel to *GL*, neither the *H* nor *V* piercing point is available for determining the line. The line is parallel to *GL*, however; hence its position may be determined by reference to the profile plane or by the method of Case 5. The intersection of the profile traces of the planes is the profile piercing point of the required line, Fig. 24(*d*).

5. If either pair of traces does not intersect within limits of drawing, or all traces meet in the same point on *GL*, two methods of solution are applicable.

**Solution** (*a*):

An auxiliary plane, parallel to *V* (or *H*), intersects the given planes in lines which intersect in a point of the required line of intersection. This point, with the available piercing point, determines the required line, Figs. 24(*e*) and 24(*f*).



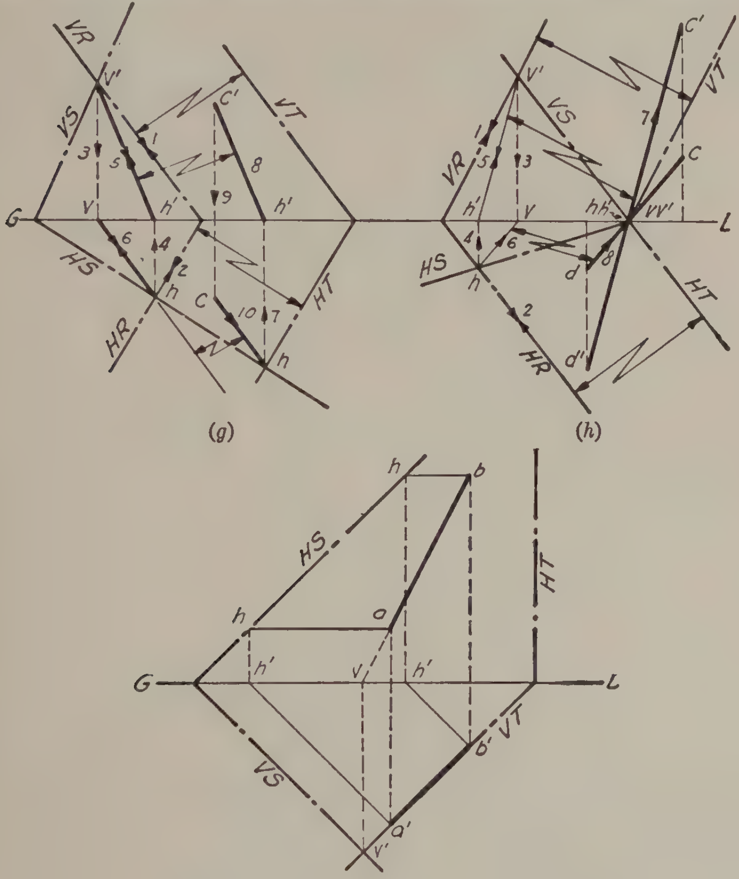
(c) (d)

(e) FIG. 24. (f)

**Solution** (*b*): (When one point on the intersection has been found.)

If in any convenient location an auxiliary plane parallel to one of the given planes is constructed, its line of intersection with the other given plane is parallel to the required line of intersection. The required line is determined by the one point and this direction, Figs. 24(*g*) and 24(*h*).

6. If one of the planes is perpendicular to a coordinate plane, the trace on that coordinate plane is the corresponding projection of the line of intersection, and the other projection may be found by means of horizontals or verticals, Fig. 24(*i*).



(*g*) (*h*)

(*i*)

Fig. 24.

66. Problem 11. **To find the projection of a given line on a given plane.**

**Analysis:**

The projection of any point on any plane is the foot of the perpendicular dropped from the point to the plane.

The projection of any straight line on any plane is a straight line comprising the projections of all points of the given line. It is defined by any two of its points.

**Solution:**

If through any two points of the given line, perpendiculars to the given plane are passed, the line connecting the piercing points of these perpendiculars in the given plane is the required projection.

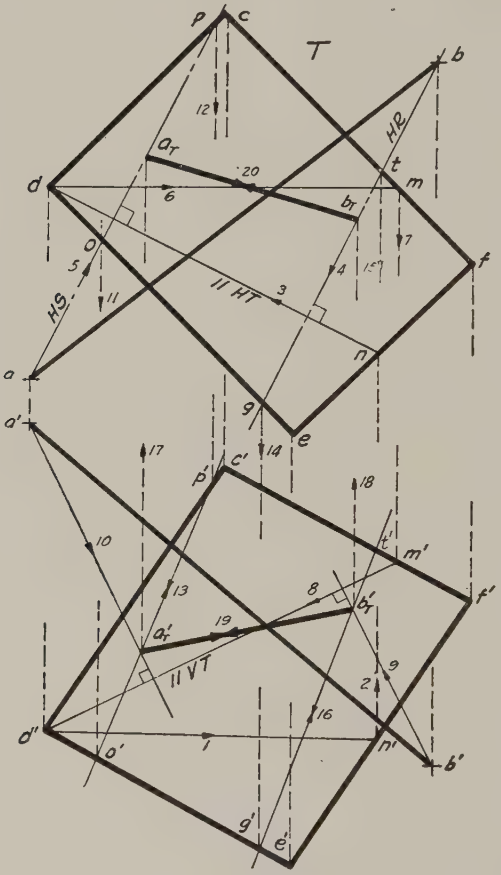


FIG. 25(*a*).

**Construction** (*a*):

Given: Plane *T* (*CDEF*) and line *AB*, Fig. 25(*a*).

Required: Projection of *AB* on *CDEF*.

Perpendiculars from *A* and *B* to *CDEF* are described by projections which are perpendicular, respectively, to the *H* projection of a *horizontal* $DN$ and to the $V$ projection of a *vertical* $DM$ (*1,10*). $H$ projecting planes $S$ and $R$ through these perpendiculars intersect plane $CDEF$ in lines $OP$ and $GT$ (*11,16*). These lines of intersection, $OP$ and $GT$, in turn intersect the perpendiculars in points $A_T$, $a'_Ta_T$ (*10,13,* and *17*) and $B_T$, $b'_Tb_T$ (*9,16* and *18*), the projections of $A$ and $B$ on $CDEF$. $A_TB_T$ (*19,20*) is therefore the required projection of line $AB$ on plane $CDEF$.

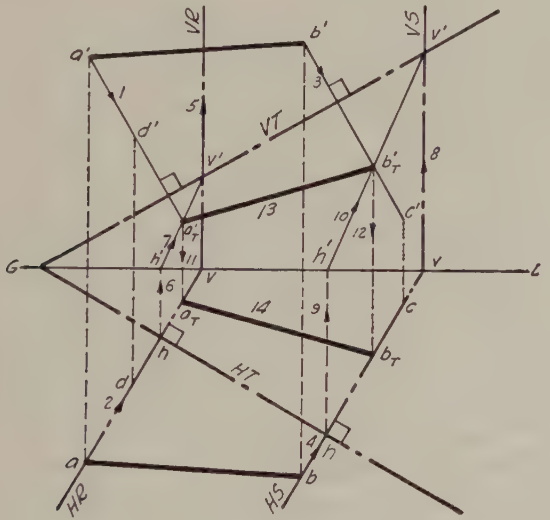


FIG. 25 (*b*).

**Construction** (*b*):

Given: Plane $T$ and line $AB$, Fig. 25(*b*).

Required: Projection of $AB$ on $T$.

Perpendiculars to plane $T$ are drawn from points $A$ and $B$ (*1,4*), [Article 54]. Through these are passed $H$ projecting planes, $R$ and $S$ (2/5 and 4/8). The lines of intersection of $R$ and $S$ with $T$ are $VH$ (*6,7*) and $VH$ (*9,10*). The perpendiculars from $A$ and $B$ intersect these lines of intersection at points $A_T$ and $B_T$ (*11,12*), which are the projections of $A$ and $B$ on $T$. Line $A_TB_T$ (13,14) is the required projection of $AB$ on plane $T$.

## SUMMARY OF RELATIONS OF POINTS, LINES, AND PLANES

**67.** It may be well to summarize here certain relations concerning points, lines, and planes, some of which may be determined by inspection of the given projections and traces, others of which cannot.

1. For descriptions of the positions of points, lines, and planes relative to the planes of projection, see:

(*a*) Alphabet of a point, Article 21.
(*b*) Alphabet of a line, Article 25.
(*c*) Alphabet of a plane, Article 38.

2. (*a*) If a point lies on a line, its $H$, $V$, and $P$ projections must lie on the corresponding projections of the line.

(*b*) In general, a point lies on a line if any **two** projections of the point lie on the corresponding projections of the line. However, *if the line is parallel to one of the planes of projection,* the projections of the point and line upon that plane must be found to determine whether the point lies on the line or not.

(*c*) *If a point and line lie in a given oblique plane,* the point lies on the line if any **one** of its projections lies on the corresponding projection of the line.

3. (*a*) If two lines intersect, their corresponding $H$, $V$, and $P$ projections must intersect, and the common points of their $H$, $V$, and $P$ projections must be the corresponding projections of the **same** point.

(*b*) In general, two lines intersect if any **two** pairs of their corresponding projections intersect at the corresponding projections of the **same** point. However, *if one of the lines is parallel to one of the planes of projection,* the projections upon that plane must be found to determine whether the point lies on that line or not.

(*c*) *If two lines lie in a given oblique plane,* the lines intersect if any **one** pair of their corresponding projections intersect.

4. (*a*) If two lines are parallel, their corresponding $H$, $V$, and $P$ projections must be parallel.

(*b*) In general, two lines are parallel if any **two** pairs of their corresponding projections are parallel. However, *if the* *lines are parallel to one of the planes of projection*, their projections upon that plane must be found to determine whether the lines are parallel or not.

(*c*) *If two lines lie in a given oblique plane*, the lines are parallel if any **one** pair of their corresponding projections are parallel.

5. (*a*) If a line lies in a plane, it must either intersect or be parallel to any other line in that plane. Lines lying in the same plane are said to be **coplanar lines.**

(*b*) If two lines are nonparallel and nonintersecting, they cannot lie in the same plane. Such lines are said to be **noncoplanar** or **skew lines.**

(*c*) If a line lies in a plane, its *H*, *V*, and *P* piercing points must lie on the corresponding traces of the plane.

(*d*) If a line lies in an oblique plane and is parallel to one of the planes of projection, it must be parallel to the corresponding trace of the plane.

(*e*) A point lies in a plane if it lies on any line in that plane.

(f) All points and lines lying in a plane which is perpendicular to one of the planes of projection must have their projections upon that plane lying in the same straight line, i.e., in the corresponding trace of their plane.

6. (*a*) A line is parallel to a plane if it is parallel to any line in that plane.

(*b*) If a line is parallel to a plane, its *H*, *V*, and *P* projections are not in general parallel to the corresponding traces of the plane.

7. (*a*) Two planes are parallel if any two **intersecting** lines in one plane are parallel to the other plane.

(*b*) If two planes are parallel, their corresponding *H*, *V*, and *P* traces must be parallel.

(*c*) Two planes are parallel if any **two intersecting** traces of one plane are respectively parallel to the corresponding traces of the other plane. However, if **two parallel** traces of one plane are parallel to the corresponding traces of another plane, their traces upon the third plane of projection must be found to determine whether the planes are parallel or not.

8. (*a*) Two lines **intersecting** at 90 degrees are said to be **perpendicular.**

(*b*) Two **intersecting** lines are perpendicular if one of the lines lies in a plane perpendicular to the other.

(*c*) If a line lies in a plane perpendicular to another line, but does not intersect the other line, the lines are said to be **rectangular lines.**

(*d*) If two lines are perpendicular, their corresponding *H*, *V*, and *P* projections are not in general perpendicular. However, *if one of the lines is parallel to one of the planes of projection,* their projections upon that plane must be perpendicular.

9. (*a*) A line is perpendicular to a plane if it is perpendicular to two intersecting lines in that plane at their point of intersection.

(*b*) If a line is perpendicular to a plane, its *H*, *V*, and *P* projections must be perpendicular to the corresponding traces of the plane.

(*c*) If a line is perpendicular to a plane, its *H*, *V*, and *P* projections must be respectively perpendicular to the corresponding projections of any *horizontal, vertical,* and *profile* of that plane.

(*d*) In general, a line is perpendicular to a plane if **any two** of its projections are perpendicular to the corresponding traces of the plane. However, *if the plane is represented by two parallel traces,* the projection of the line and the trace of the plane upon the third plane of projection must be found to determine whether the line is perpendicular to the plane or not.

10. (*a*) Two planes are perpendicular if one of the planes contains a line perpendicular to the other plane.

(*b*) If two planes are perpendicular, their corresponding *H*, *V*, and *P* traces are not in general perpendicular. However, *if one of the planes is perpendicular to one of the planes of projection,* their traces upon that plane must be perpendicular.

## CHAPTER V

# HIGHER RELATIONS OF THE POINT, LINE, AND PLANE

### PRINCIPLE "D"

### REVOLUTION

**68.** If lines and planes are rotated into a position coinciding with or parallel to the coordinate planes of projection, the condition of plane geometry is achieved wherein all properties of lines and points are measurable, with the conventional standard measuring devices (scales, protractors, planimeters).

**69.** Problem 12. **To revolve a given point about a given line, and to find its projections in any revolved position.**

**Discussion:**

The term "**revolution**" may be broadly defined as implying "*motion in a circular path.*" Thus, if a given point, *O*, Fig. 1, moves so that it remains in a given plane, *R*, and at a fixed distance from another given point, *C*, in that plane, point *O* will describe the circumference of a circle lying in plane *R* with center at *C* and radius *CO*; and point *O* is said to revolve about

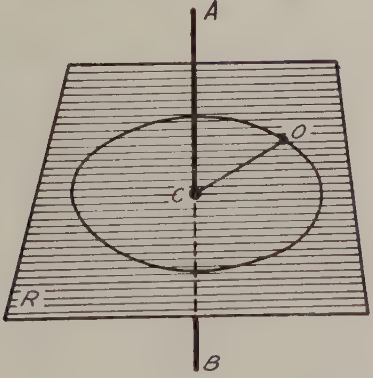


FIG. 1.

point $C$ as a center. Or, if a line, $AB$, is taken perpendicular to plane $R$ through point $C$, point $O$ is said to revolve about line $AB$ as an axis. Point $O$ is termed the **revolving point**; plane $R$, the **plane of revolution**; point $C$, the **center of revolution**; line $CO$, the **radius of revolution**; line $AB$, the **axis of revolution**; and the circle described is called the **circle of revolution.**

It follows from the above that the problem of revolving a given point about a given axis involves only the determination of a certain definitely fixed circle. It will be recalled from plane geometry that the determination of a circle involves the fixing of the following three characteristics of the circle: its plane, its center, and its radius. When these three elements are fixed, the circle, and every point on its circumference, are determined. Since the plane of revolution contains the revolving point and is perpendicular to the axis of revolution, the plane is determined, and knowing that the center of revolution lies on the axis of revolution and in the plane of revolution, the center is determined. Since the revolving point is one point on the circumference of the circle of revolution the radius of the circle of revolution is determined.

## APPLICATIONS OF REVOLUTION

**70.** The problem of revolving a point about a straight line, which we have called the Principle of Revolution, finds many applications in the further problems of descriptive geometry and is indispensable to the determination of values.

The determination of values (magnitudes) by the comparative process of measurement through the use of scales, a protractor, or planimeter requires that the distance, angle, or area to be measured be described in true or known scale value in the plane in which the measuring instruments are used. This implies that the value to be measured was drawn in the plane of measurement, or that it has been projected upon or placed in that plane in such manner that the projection or transferred description has the same value as the original. This can only be true when the *value* (*line, angle,* or *area*) is made *parallel* to the *plane of measurement* or the *plane of the value* is made to *coincide with the plane of measurement.* There is no other process than this, no matter what name it is given—revolution, auxiliary projection, value planes, etc.

**71.** Problem 13. **To revolve a given point about a vertical line and find its projections when it is the same distance from *V* as the line.**

**Note:** This problem might be stated in more general terms as **"To revolve a given point about a line which is perpendicular to a plane of projection and determine its projections when it is in any desired position on the circle of revolution."** It is felt, however, that the desired end of determining the distance between two points can be attained by stating the problem in more precise and restricted terms. One axis is therefore selected, preferably perpendicular to the *H* plane of projection, and stated as a "*vertical*" line because of the simple and direct connotation of that word.

**Analysis:**

The path of a point which revolves about a straight line is a circle.

The plane of the path is perpendicular to the line.

The center of the circular path is on the line.

The radius of the path is the distance from the point to the line.

The relative distances from points and vertical lines to *V* are shown by their horizontal projections.

**Solution:**

If a plane is passed through point *O* perpendicular to the given line *AB*, it will intersect the line in a point, *C*, which is the center of the circular path in which *O* travels. The distance from *O* to this point *C* is the radius of the circle.

**Construction:**

Given: Point *O* and the vertical line *AB*, Figs. 1 and 2.

Required: To revolve *O* about *AB* until it is the same distance from *V* as *AB*.

Since line *AB* is vertical, plane *R* of the path of *O* is horizontal. Its vertical trace *VR* is therefore horizontal and passes through $o'$. Plane *R* intersects line *AB* at point *C*, which is the center of the path of *O*. The radius of the circle is *oc*, and

$c$ is the center of the $H$ projection of the circle, which in this case is identical in size and shape with the circle itself. Point $O$ will be the same distance from $V$ as $AB$ when its $H$ projections lie on a horizontal line through $ab$; that is, at $o_1$ and $o_2$, (4). The $V$ projections of all points of the circle lie on $VR$; hence the $V$ projections of $O_1$ and $O_2$ are at $o_1'$ and $o_2'$ (*5*, *6*).

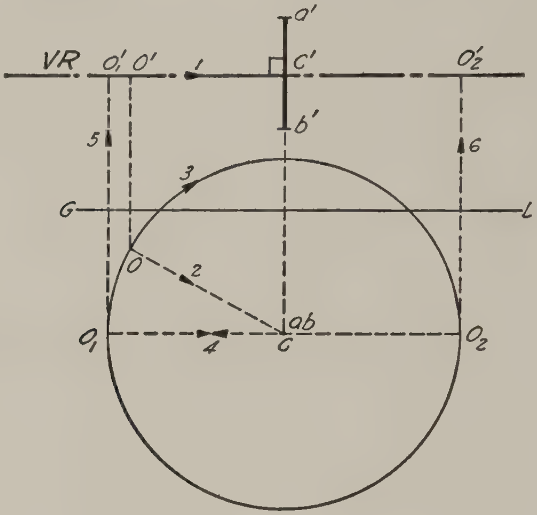


FIG. 2.

**72.** Problem 14. **To find the true length of a given line.**

**Analysis** (Method A):

The portion of a line between any two points shows in its true length if the line lies in the plane of measurement or is parallel to it.

A line may be rotated into a position parallel to a plane of measurement, e.g., $V$, by revolving one extremity about a vertical line through the other extremity until they are the same distance from $V$ [Article 71].

**Solution:**

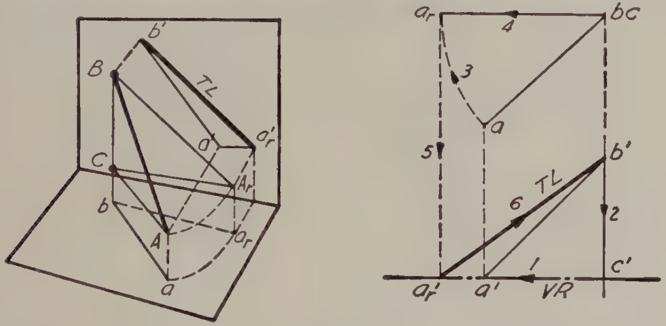
If one of the two points defining the line, Fig. 3(*a*), is revolved about the $H$ projecting line of the other until they are the same distance from $V$, the line will have been revolved into a position parallel to $V$, and its true length will then be shown on the $V$ plane.

**Construction** (*a*):

Given: Line *AB*, Fig. 3(*b*).

Required: True length of *AB*.

Point *A* is to be rotated about the *H* projecting line of *B* until they are equally distant from *V*. Since the axis of revolution is vertical, the plane of rotation *R* is horizontal and its *V* trace *VR* is horizontal and passes through $a'$ (*1*). The center of rotation is at *C* and the radius is *AC*. The horizontal projection of the circle is drawn with center *c* and radius *ac* (*3*). The arc (*3*) from *a* to $a_r$ on the horizontal line (*4*) is all that is required, for *A* is then the same distance from *V* as is *B*. The *H* projection of the line in this revolved position is $a_rb$ (*5*,*6*), and as the *H* projection of *A* moves to $a_r$ the *V* projection $a'$ moves along *VR* to $a_r'$. The *V* projection of *AB* when it has been rotated about *BC* until it is parallel to *V* is $a_r'b'$ (*6*). It therefore provides the required true length of *AB*.



(*a*) (*b*)

Fig. 3.

**Solution** (*b*), (Method B):

If the quadrilateral formed by the line, the projecting lines (e.g., *H*) of two of its points, and its *H* projection are revolved about the *H* projection into *H*, Fig. 4(*a*), the quadrilateral will show in its true size and form and the line in its true length.

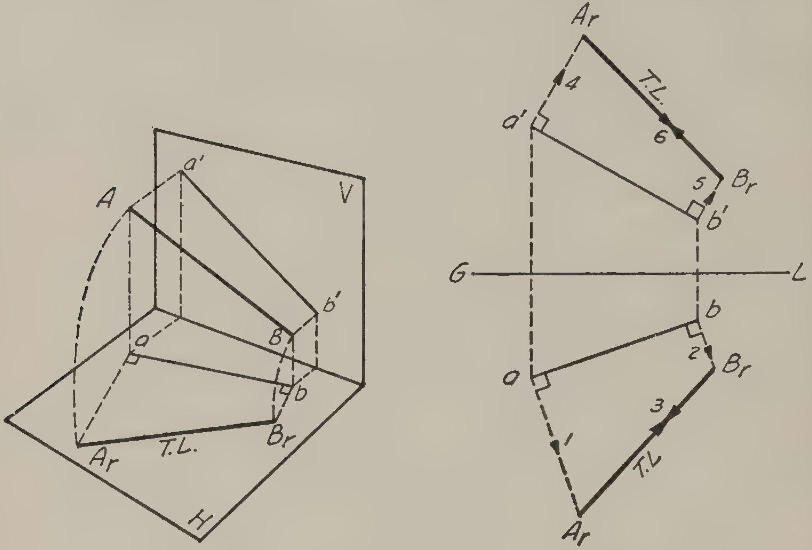
**Note:** Method B is given here in this manner to associate it with Method A. It is explained again in Article 79.

**Construction** (*b*):

Given: Line *AB*, Fig. 4(*b*).

Required: True length of *AB*.

The projecting lines from points $A$ and $B$ make an angle of 90 degrees with the respective projections $ab$ and $a'b'$ of the line. The revolved true size and shape quadrilaterals can therefore be constructed by erecting perpendiculars to the $H$ projection $ab$ at $a$ and $b$ (*1*,*2*) and measuring on them from $a$ and $b$ the distances, respectively, from $a'$ and $b'$ to $GL$ to locate $A_r$ and $B_r$. $A_rB_r$ is the required true length. In similar fashion $A_rB_r$ may be found by revolution about $a'b'$ into the $V$ plane.



(*a*) (*b*)

FIG. 4.

**73.** Problem 15. **To find the distance from a given point to a given plane.**

**Analysis:**

The distance from a point to a plane is measured along a perpendicular from the point to the plane.

The condition of perpendicularity can be seen with a *horizontal* and *vertical*, or with the $H$ and $V$ traces.

The distance is the length of the portion of the perpendicular from the given point to its piercing point in the plane.

The distance will show in its true value if the perpendicular is revolved into or parallel to a plane of measurement.

**Solution:**

If through the given point *O*, Fig. 5(*a*), a perpendicular to the given plane is passed, the true length of the portion of the perpendicular between the given point and the piercing point *P* will be the required distance.

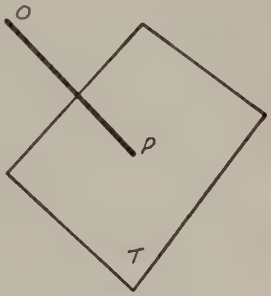


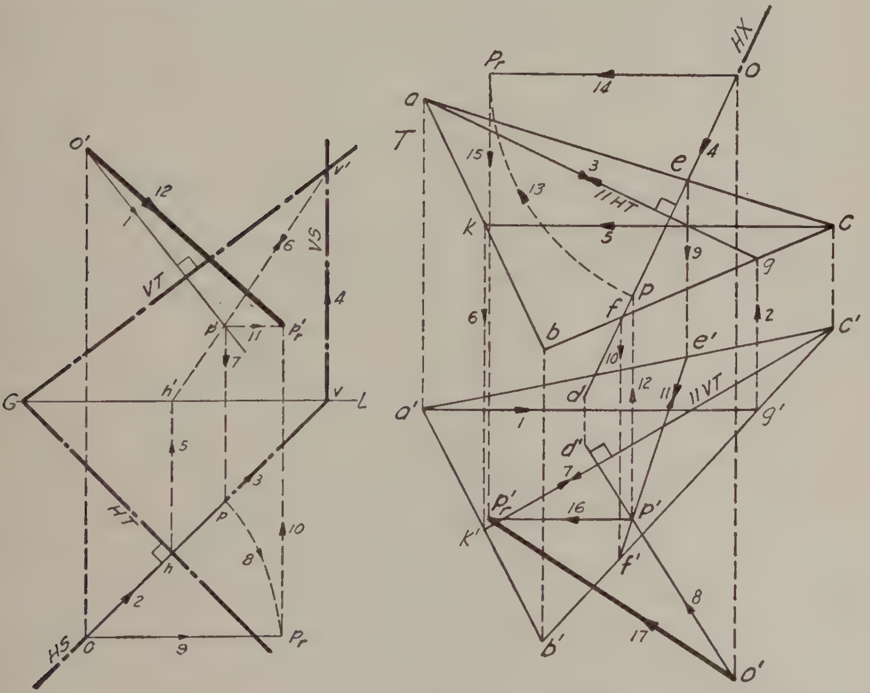
FIG. 5(*a*).

**Construction** (*a*):

Given: Plane *T* and point *O*, Fig. 5(*b*).

Required: Distance from *O* to *T*.

A perpendicular to plane *T*, Fig. 5(*b*), is drawn through *O* (*1*,*2*), [Article 54]. The *H* projecting plane *S* of the perpendicular intersects plane *T* in line *VH*(*3*,*6*,) which intersects the perpendicular in point *P* (*7*), its piercing point. *OP* is revolved about the *H* projecting line of point *O*



(*b*) (*c*)

FIG. 5.

until it is parallel to $V$ (*8,12*). The $V$ projection $o'p'_r$ in this revolved position is the true length of $OP$ and the required distance [Article 72].

**Construction** (*b*):

Line $OD$, perpendicular to plane $T$, is passed through $O$, Fig. 5(*c*). Its projections, $od$ and $o'd'$, are perpendicular respectively to the $H$ projection of a *horizontal* (*1,4*) and to the $V$ projection of a *vertical* (*5,8*) of plane $T$. An $H$ projecting plane through $OD$ intersects plane $T$ in the line $EF$ (*9,11*). $EF$ intersects the perpendicular $OD$ in the required piercing point $P$ (*12*). $OP$ is revolved about the $H$ projecting line of $O$ until it is parallel to $V$ (*13,17*). The true length of the line $OP$ and the required distance from $O$ to plane $T$ are $o'p'_r$ (*17*).

74. Problem 15: Example 1. **To find the distance between two parallel planes.**

**Analysis:**

The distance between parallel planes is measured along a common perpendicular to them.

The true length of the portion of the perpendicular between its piercing points in the two planes is the distance between the planes.

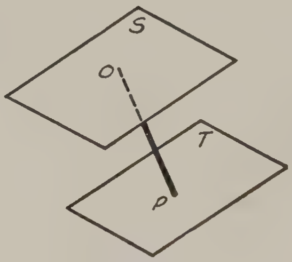


Fig. 6(*a*).

**Solution:**

If through any point $O$ in one plane $S$, Fig. 6(*a*), a perpendicular to plane $T$ is passed, the true length of the portion of the perpendicular from the assumed point $O$ in $S$ to its piercing point $P$ in $T$ will be the required distance.

**Construction:**

Given: Parallel planes $S$ $(OMN)$ and $T$ $(ABC)$, Figs. 6(*b*) and (6*c*).

Required: Distance between $S$ and $T$.

A perpendicular to plane $T$ is passed through any convenient point $O$ in plane $S$. Its projections $od$ and $o'd'$ are perpendicular respectively to the $H$ projection of a *horizontal* (*1,4*) and to the $V$ projection (*5,8*) of a *vertical* of plane $T$. An $H$ projecting plane $X$ through $OD$ intersects plane $T$ in the line $EF$ (*10,12*).

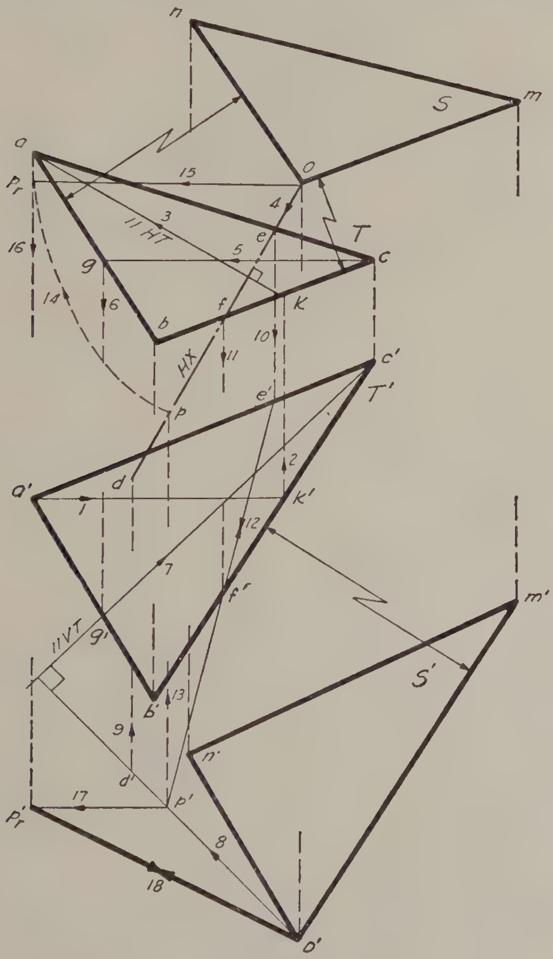


FIG. 6(*b*).

*EF* intersects *OD* in the required piercing point *P* (*13*). The true length of *OP* is $o'p'_r$ (*14,18*).

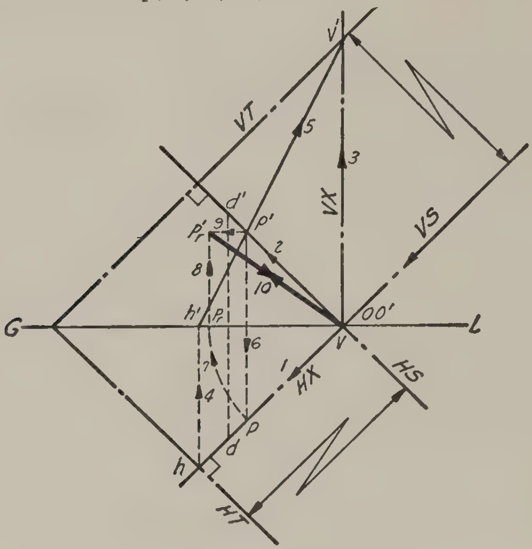


FIG. 6(*c*).

### 75. Problem 16. **To find the distance from a given point to a given line.**

**Analysis:**

The distance from a given point to a given line is measured along a perpendicular from the point to the line.

A plane through the given point perpendicular to the given line contains the perpendicular line along which the desired distance must be measured.

The line connecting the given point with the piercing point of the given line in the perpendicular plane is perpendicular to the given line.

**Solution:**

If a plane, *T*, is passed through the given point *O* perpendicular to the given line *AB*, Fig. 7(*a*), the distance from the given point to the point *P* in which the given line pierces this plane will be the required distance.

**Construction:**

Given: Point *O* and line *AB*, Figs. 7(*b*) and 7(*c*).

Required: Distance from *O* to *AB*.

Plane *T*, Fig. 7(*b*), is passed through point *O* perpendicular to line *AB* (*1,6*), [Article 56].

The *H* projecting plane *X* of *AB* intersects plane *T* in line *EF* (*7,11*), Fig. 7(*b*). *EF* intersects *AB* in the desired point *P* and *OP* is the required perpendicular line from *O* to *AB* (*12–14*). The true length of *OP* is $o'p'_r$ (*15,18*).

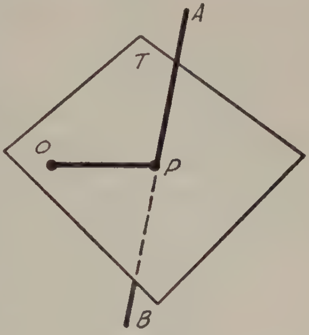
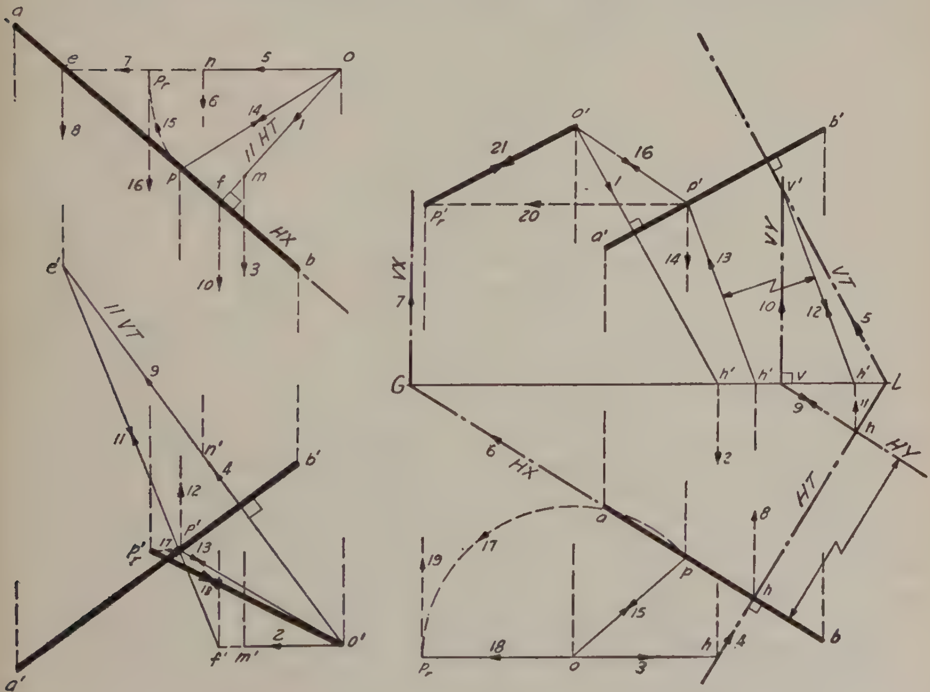


Fig. 7(*a*).

Plane *T*, Fig. 7(*c*), is passed through point *O* perpendicular to line *AB* (*1,5*). The *H* projecting plane *X* of *AB* intersects plane *T* in line *HP* (*8* and *13,14*) parallel to the intersection *VH* (*9,12*) of planes *Y* and *T*. The perpendicular from *O* to *AB* is then *OP* (*15,16*) and its true length is $o'p'_r$ (*17,21*).



(*b*)
(*c*)

Fig. 7.

76. Problem 16: Example 1. **To find the true distance between two given parallel lines.**

**Analysis:**

The distance between two parallel lines is measured along a common perpendicular line.

The distance is the true length of the portion of the perpendicular between its intersections with the given lines.

A plane through any point of the one line and perpendicular to both lines intersects the second line in the second required point. The line joining these points is the required perpendicular.

**Solution:**

If a plane, $T$, is passed through any point $O$ on line $CD$ and perpendicular to line $AB$, Fig. 8, the distance from this point to point $P$, in which the second line pierces this plane, will be the required distance.

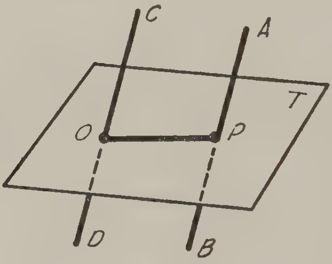


Fig. 8.

**Construction:**

Given: Parallel lines $AB$ and $CD$, Fig. 8.

Required: Distance between $AB$ and $CD$.

Having assumed point $O$ on $CD$, the construction is identical with that in Figs. 7($b$) and 7($c$), [Article 75], for measuring the distance from a point to a line.

77. Problem 17. **To find the projections and true length of the common perpendicular to two non-intersecting and non-parallel lines.**

**Analysis:**

If a line, $CD$, is parallel to a plane, $T$, it is parallel to its projection $C_T D_T$ on that plane, Fig. 9($a$).

A perpendicular to plane $T$ at any point on the projection $C_TD_T$ will intersect the given line $CD$ and be perpendicular to it.

If a line, $AB$, in the plane intersects the projection $C_TD_T$, the perpendicular to plane $T$ at this point $O$ is perpendicular to both $CD$ and $AB$.

**Solution:**

If through the line $AB$ a plane $T$ is passed, parallel to $CD$, Fig. 9($a$), and the projection of $CD$ on plane $T$ is found, the perpendicular to plane $T$, at the point in which the projection of $CD$ on $T$ intersects $AB$, will intersect $CD$ and is the required common perpendicular.

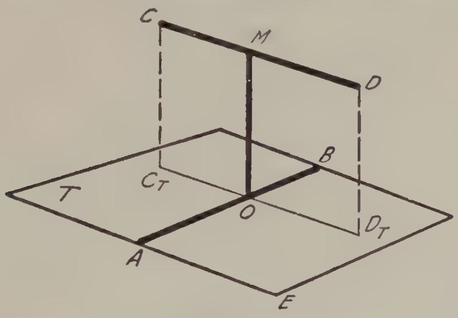


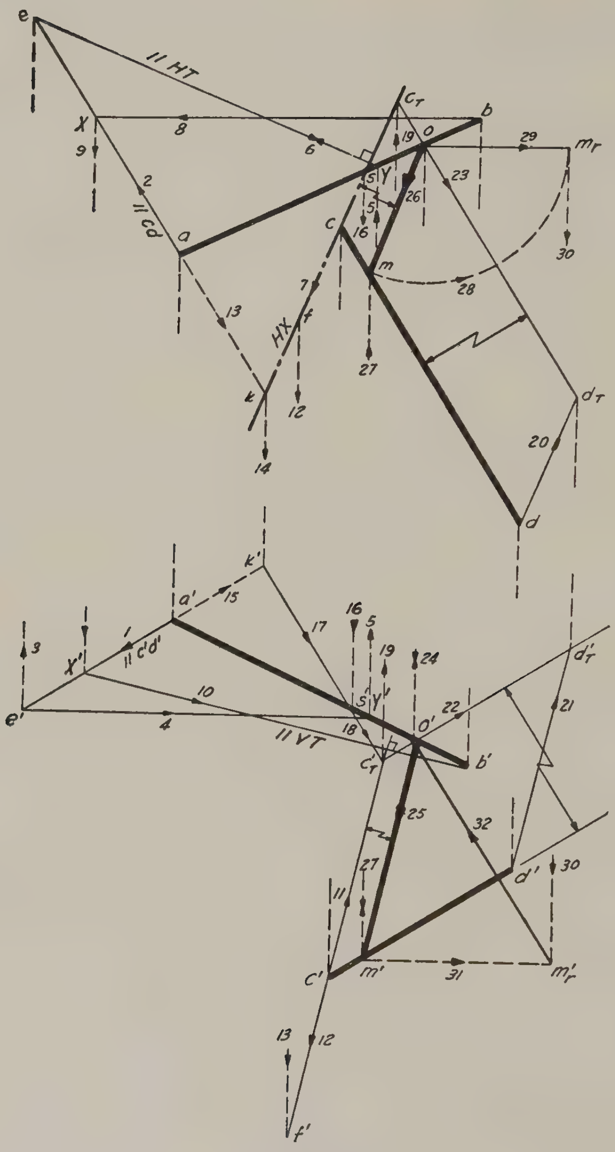
FIG. 9($a$).

**Construction** ($a$):

Given: Lines $AB$ and $CD$, Fig. 9($b$).

Required: Common perpendicular to $AB$ and $CD$.

A line, $AE$, is passed through $A$ parallel to $CD$ (*1,3*). The plane $EAB$ ($T$) is thus parallel to $CD$. A perpendicular, $CF$, to plane $T$ is passed through point $C$; $cf$ is perpendicular to the $H$ projection of a *horizontal* (*4,7*) and $c'f'$ to the $V$ projection of a *vertical* of plane $T$ (*8,13*). The $H$ projecting plane $X$ of $CF$ intersects $T$ in line $SK$ (*14,17*), which intersects $CF$ in point $C_T$ (*18,19*), the piercing point of $CF$ in $T$. The projection of $CD$ on $T$ ($C_TD_T$) is parallel to $CD$, because $CD$ is parallel to $T$. The projections of $C_TD_T$ are hence parallel to $cd$ and $c'd'$ (*22,23*). $C_TD_T$ intersects $AB$ in point $O$ (*24*), which is one point on the common perpendicular. A perpendicular to $T$ through $O$ is parallel to $CF$ (*25,26*) and intersects $CD$ in point $M$ (*27*), the other point on the common perpendicular. The true length of the common perpendicular $OM$ is $o'm'_r$ (*28,32*).


e
// HT
X
8
9
2
// cd
a
6
c_T
b
19
o
29
m_r
s
Y
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26
23
c
16
m
7
13
HX
f
28
30
27
k
12
14
d_T
20
d
k'
a'
15
17
16
5
19
24
1
// c'd'
X'
3
e'
10
4
s'
Y'
18
// VT
O'
22
d'_T
21
c'_T
b'
25
32
27
11
30
d'
c'
m'
31
m'_r
13
12
f'

Fig. 9(b).

**Construction** (*b*):

Given: Lines *AB* and *CD*, Fig. 9(*c*).

Required: Common perpendicular to *AB* and *CD*.

*AE* is passed through point *A* parallel to *CD* (*1,3*). Plane *T* of *AB* and *AE* is the required plane through *AB* parallel to *CD* (*4,14*). Lines *CF* and *DG* are drawn perpendicular to plane *T* (*15,16*) and (*21,22*). Their piercing points in plane *T* are $C_T$ (*17,20*) and $D_T$ (*23,26*). The projection of *CD* on *T* is hence $C_TD_T$ (*27,28*). From point *O* (*29*), at which this projection $C_TD_T$ intersects *AB*, the required perpendicular to plane *T* is erected (*30,31*); it intersects *CD* at *M* (*32*). $o'm'_r$ (*33,37*) is the true length of the common perpendicular.

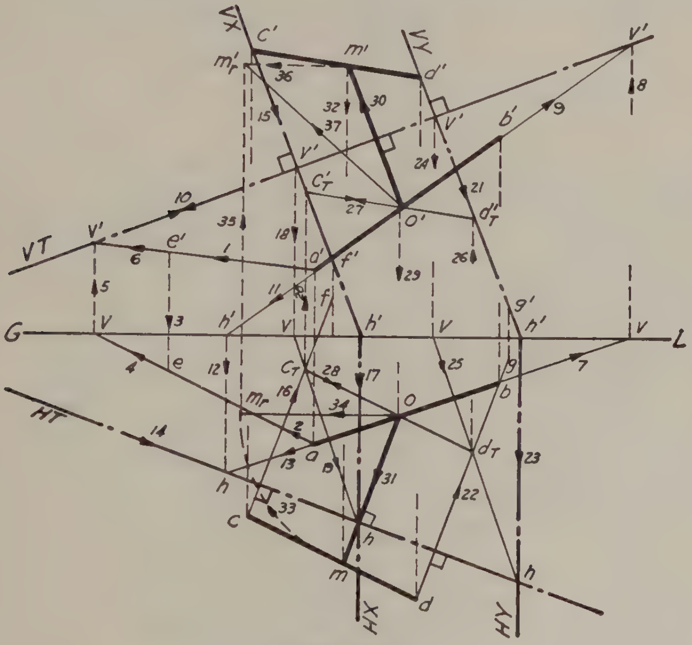


FIG. 9(*c*).

## CHAPTER VI

# APPLICATIONS OF REVOLUTION IN MEASUREMENT OF PLANE FIGURES

### PRINCIPLE "D"

### REVOLUTION

**Angles—True shape of plane figures—Revolved positions.**

78. Problem 18. **To revolve a given point, *O*, about a given horizontal line, *AB*, and find the points in which the path of *O* pierces:**

1. The horizontal plane through *AB*.
2. Any vertical plane; e.g., *V* or *P*.

**Note:** Like Problem 13, this problem may be stated more generally as **" To revolve a given point about a given line which is parallel to a plane of projection and determine its projections when it is in any desired position on the circle of revolution."** Again it is felt that the desired end of determining the true values of angles, true shapes of plane figures, and revolved positions can best be attained by stating the problem in simple and precise terms of easy connotation, that is, a "*horizontal*"

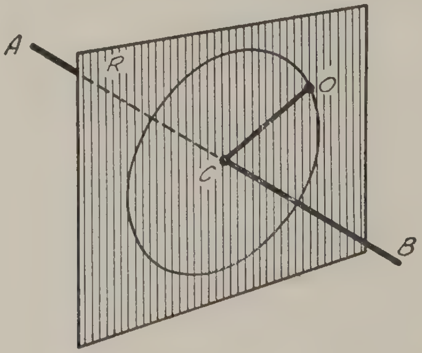


FIG. 1(*a*).

line as an axis. The application of the method to lines parallel to *V* and *P* will be obvious.

**Analysis:**

The path of point *O* is a circle.

The plane of the circular path is perpendicular to the line *AB*.

The center *C* of the circle is the point in which *AB* pierces the plane of the circle.

The radius is the distance from *O* to *C*.

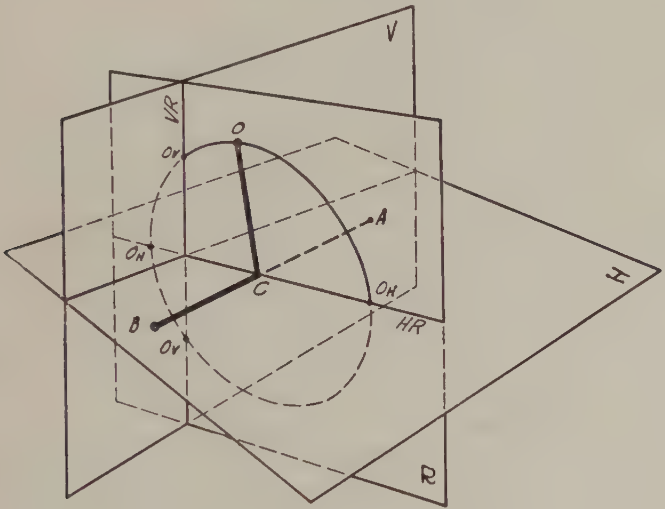


FIG. 1(*b*).

The circular path of *O* will pierce planes *H* and *V* in points on the lines of intersection of the plane of the circle with *H* and *V*.

**Solution:**

If a plane, *R*, is passed through point *O* perpendicular to line *AB*, Fig. 1(*b*), the center of the circular path is the point *C*, in which *AB* pierces *R* and the radius is the distance from *O* to *C*. The circle will pierce *H*, *V*, and *P* in points on the lines of intersection of *R* with *H*, *V*, and *P* at a distance from *C* equal to the radius.

**Discussion:**

Since the axis of revolution is horizontal, it will be convenient to locate the *H* plane so as to contain the line. That is, *GL* will coincide with the *V* projection of the line. The plane of revolution, being perpendicular to the axis, will be perpendicular also to *H*. Since the axis lies in *H*, the point in which it pierces the plane of revolution, *R*, will be a point in *H*, and on the *H* trace of the plane. With its center in *H*, one half of the circle will lie above and the other half below *H*, with one diameter in the *H* trace of the plane, Fig. 2.

The plane of revolution being thus an *H* projecting plane, all points on the circle will have their *H* projections in its *H* trace.

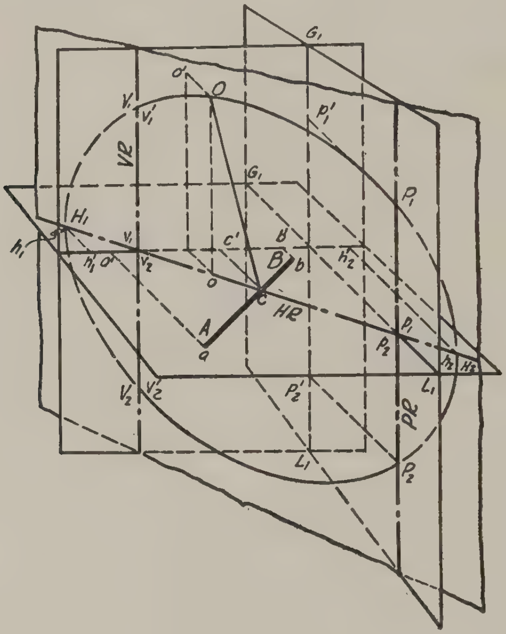


Fig. 2.

As the given point moves around the circle, its *H* projection moves along the *H* trace, always *perpendicularly below or above the moving point;* i.e., the *H* projection of the circle is a straight line equal to a diameter, and lying in the *H* trace of the plane. The *V* projection of the circle will in general be an ellipse, but it will not be necessary to construct this ellipse for the solution of any problem. Instead, the plane of revolution will be revolved about its *H* trace into *H*, in which position the circle will show in its true size and shape in the *H* projection, Fig. 3(*a*). The *H* projecting line of any point on the circle will, in this revolved position, fall perpendicular to the $H$ trace at the $H$ projection of the point. The length of this perpendicular, being the distance of the point above or below $H$ on the original position of the circle, is equal to the distance of the $V$ projection from $GL$. If the position of the axis and length of the radius are such that the circle in its original position pierces $V$ and $P$, the revolved positions of these piercing points are found at the points in which the revolved circle intersects the revolved positions of the $V$ and $P$ traces. Since the plane is revolved about its $H$ trace into $H$, this trace and the points in which it is intersected by the circle remain in their original positions.

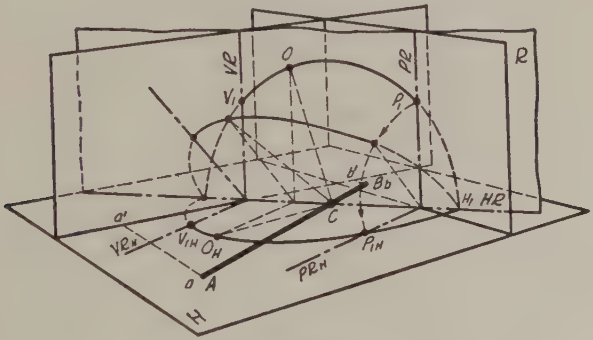


Fig. 3(a).

Since the $V$ and $P$ traces of the plane of revolution are perpendicular to its $H$ trace at the points in which the $H$ trace intersects $GL$ and $G_1L_1$, respectively, Figs. 3(a) and 3(b), they will fall perpendicular to the $H$ trace at these points when the plane is revolved into $H$.

The plane in its original position passes through all four quadrants, the part in any quadrant being bounded by one of the four right angles formed by the intersection of the $H$ and $V$ traces. In the revolved position these four right angles are shown in their true size between the $H$ trace and the revolved position of the $V$ trace. The quadrant in which the given point lies is known, and every other point whose revolved position is in the same angle between the $H$ and $V$ traces must lie in that same quadrant. The quadrants which contained the other three right angles may be determined by comparing their positions relative to the $H$ and $V$ traces in the revolved position with that of the known quadrant in which the given point lies. Thus, if the given point is in the first quadrant, in the revolved positions all points on the same side of the $V$ trace as the given point but on the opposite side of the $H$ trace are in the fourth quadrant, etc.

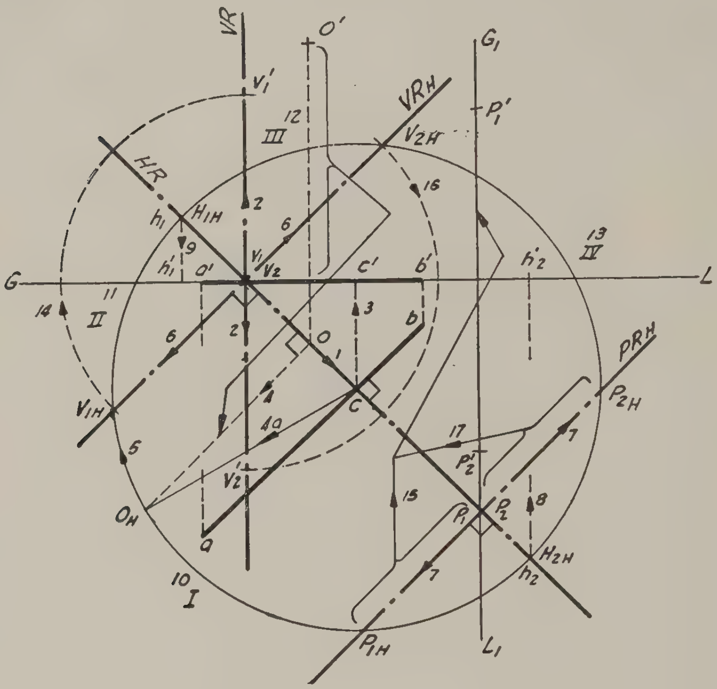


FIG. 3(b).

**Construction:**

Given: Point $O$ and horizontal line $AB$ through which the $H$ plane of projection is passed, Figs. 2, 3(*a*), and 3(*b*).

Required: The points in which the circular path of $O$ pierces the $H$ plane and the two vertical planes $V$ and $P$ as it revolves about $AB$ as an axis.

Plane $R$ is passed through $O$ perpendicular to $AB$. Since $AB$ lies in $H$, plane $R$ is vertical. Its $H$ trace therefore passes through the $H$ projection of $O$ and is perpendicular to the $H$ projection of $AB$ (*1*). The $V$ trace is vertical (*3*) and in space is also perpendicular to $HR$. In space the $P$ trace is likewise perpendicular to $HR$ at the point in which the latter intersects $G_1L_1$. But since the solution is to be made in $H$ and $V$ projections, the usual $P$ projection of the $P$ trace will not be shown.

By inspection, the center of revolution is on $HR$ at $C$ (*2*), its intersection with $AB$.

Having determined the plane and center of revolution, the plane is rotated about its $H$ trace into the $H$ plane. In this rotation the given point may be made to fall on either side of the $H$ trace. $Oo$, the $H$ projecting line of $O$, is perpendicular to $HR$ and remains so in the rotation of plane $R$ into $H$. Point $O$ therefore falls at $O_H$ on the revolved position of this perpendicular (*3*) and at a distance from $HR$ equal to the distance of $o'$ from $GL$. The radius of revolution is shown in its true length at $cO_H$ (*4*) and the circle is drawn in its true size and shape (*5*). The $V$ and $P$ traces fall at $VR_H$ and $PR_H$ (*6,7*) perpendicular to $HR$ at the points in which $HR$ intersects $GL$ and $G_1L_1$. The revolved positions of the $V$ and $P$ piercing points of the circle are at $V_{1H}$, $V_{2H}$, and $P_{1H}$, $P_{2H}$, where the circle intersects these traces in the revolved positions. Since the plane was revolved about its $H$ trace, that trace $HR$ and the points $H_1$ and $H_2$ in which the circle intersects it remain in their original positions, as $H_{1H}$ and $H_{2H}$.

Having determined the six piercing points, on the revolved position of the circle, the $H$ projection of each is in $HR$ at the foot of the perpendicular from the point on the circle. The $H$ projections of $H_1$ and $H_2$ coincide with $H_{1H}$ and $H_{2H}$, and since these are points in $H$ their $V$ projections $h_1'$ and $h_2'$ lie on $GL$ (*8,9*). The $H$ projections of the two points $P$ coincide at $p_1p_2$ and of the two points $V$ at $v_1v_2$. The $V$ projections of these points are above or below $GL$ at distances equal to the lengths of the respective perpendiculars. The question as to whether $v_1'$ and $p_1'$ should be above or below $GL$, however, must now be determined.

Points which on the original position of the circle are above $H$ will all have their revolved positions on the same side of $HR$ as $O_H$, since $O$ is given above $H$; points on the opposite side of $HR$ are below $H$. Likewise, points having their revolved positions on the same side of $VR_H$ as $O_H$ must be in front of $V$ in their original positions; those on the opposite side are behind $V$. Thus it is determined that the portion of the circle in its revolved position between $V_{1H}$ and $H_{2H}$ is in the first quadrant (*10*) because point $O$ on that arc is a point which was given in the first quadrant.

Passing from this arc across the revolved $V$ trace at $V_{1H}$, the next arc (*11*) must represent the portion of the circle in the second quadrant, since passing through $V$ from the first quadrant brings one into the second. Continuing on the arc across the $H$ trace at $H_{1H}$, the next arc (*12*) must be in the third quadrant, since passing from the second quadrant through $H$ brings one into the third. Continuing through $V_{2H}$, the next arc (*13*) must be the fourth, since passing from the third quadrant through $V$ brings one into the fourth. Point $V_{1H}$ and $P_{1H}$ thus represent points above $H$ and $V_{2H}$ and $P_{2H}$ points below $H$; $v_1'$ and $p_1'$ are therefore *above* $GL$ at distances equal to $V_{1H}v_1$ and $P_{1H}p_1$ (*14,15*) and $v_2'$ and $p_2'$ *below* $GL$ equal distances (*16,17*).

**79.** Problem 18: Example 1. **To find the true length of a given line.** (Method B.) (See Article 72 for Method A.)

**Analysis:**

The true length of a line can be determined if it can be revolved into a coordinate plane about a line in that plane.

A line can be revolved into a coordinate plane about a line in that plane if the given line and the axis of revolution constitute a plane, i.e., if they intersect or are parallel.

The projections of the line qualify as axes of revolution.

The process of revolving a line into $H$ about its $H$ projection is that of revolving two points of the line about an axis in $H$ and finding the points in which their paths pierce the $H$ plane [Article 78].

**Solution:**

If each of the two points defining a line be revolved about the $H$ projection of the line until their paths pierce the $H$ plane, the line joining these piercing points is the revolved position and true length of the line.

**Construction:**

Given: Line $AB$, Fig. 4(*a*).

Required: True length of $AB$.

Planes $R$ and $S$ are passed through points $A$ and $B$ perpendicular to the $H$ projection $ab$. These planes are vertical and their $H$ traces are perpendicular to $ab$ (*1,2*). The centers of revolution are $C_1$ and $C_2$.

Point $A$ is vertically above $C_1$ at a distance equal to the distance of $a'$ above $GL$, and point $B$ is vertically above $C_2$ at a distance equal to the distance of $b'$ above $GL$. The distances $AC_1$ and $BC_2$ are the radii of revolution, and the circles of revolution will pierce $H$ in points $A_r$ and $B_r$ on the $H$ traces *HR* and *HS* (*5*,*6*).

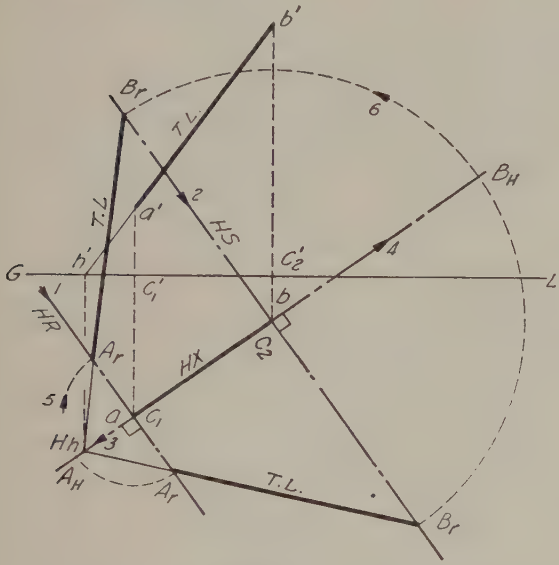


FIG. 4(*a*).

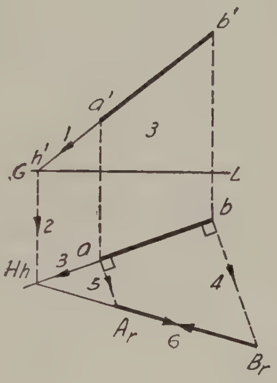


FIG. 4(*b*).

**Note.** The above procedure, which is given in the terms used in Article 78 may be stated simply as the erecting of perpendiculars to *ab* at *a* and *b* and measuring on them the distances which the corresponding *V* projections are from *GL*, Fig. 4(*b*).

**80.** Problem 18: Example 2. **To revolve a given point about a given line in *H*, into a position in a given quadrant at a given distance from *H* or *V*.**

**Analysis:**

The path of rotation is a circle.

The plane of the circular path is perpendicular to the line and hence is a vertical plane.

The center of the circle is the point in which the line pierces the plane.

The radius of the circle is the distance from the given point to the line.

The *H* projections of all points on the circle are on the *H* trace of the plane.

Distances from *H* are measured as vertical ordinates of the circle, above or below the *H* trace.

Distances from *V* are measured from the positions of the *H* projections on the *H* trace of *R* to *GL*.

**Solution** (*a*):

(Line in *H* and distance referred to *H*.)

The plane of the circle of revolution passes through the given point perpendicular to the given line. The distance from the point to the line is the radius of the circle. The distances of the moving point from *H* are equal to the vertical ordinates of the circle at the various positions considered.

**Construction** (*a*):

Given: Horizontal line *AB* and point *O*, Fig. 5.

Required: To revolve *O* about *AB* into a position $\frac{1}{2}$ inch below the horizontal plane through *AB*.

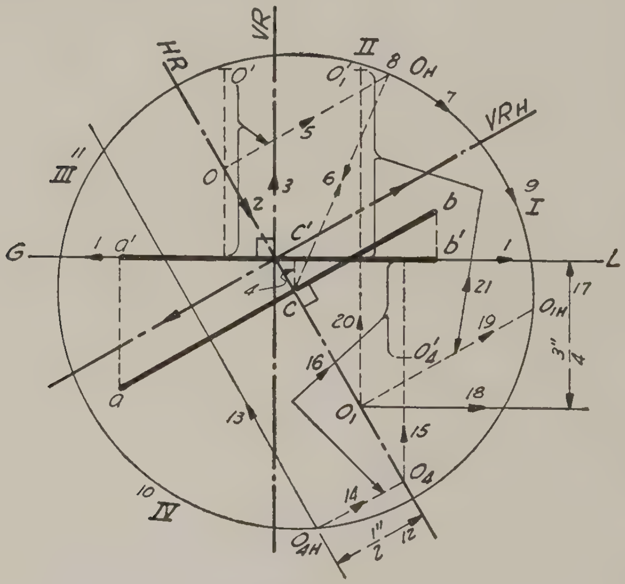


Fig. 5.

The horizontal plane through *AB* is indicated by extending *a′b′* as the ground line (*1*). Plane *R* is passed through *O* perpendicular to *AB* (*2*,*3*). The center of the circle of revolution is *C* (*4*). Plane *R* is revolved about *HR* into *H* (*5*,*7*). The arc of the circle on which $O_H$ is found lies in the second quadrant (*8*). The arcs lying in the other quadrants are as indicated (*9,11*), [Article 78]. Since plane *R* has been revolved into *H* about its *H* trace, the distances from any points on the circumference of the circle to *HR* represent the distances which the positions of these same points were from *H* before *R* was revolved, and these are indicated by the distances which the *V* projections of the points are above or below *GL*. To find the points of the circle, which are $\frac{1}{2}$ inch below *H*, a perpendicular to *HR*, $\frac{1}{2}$ inch long, and below *HR*, is erected (*12*) and through its extremity a line is drawn parallel to *HR* (*13*). This line intersects the circle at $O_{4H}$, the revolved position of one required point. The projections of $O_4$ are $o_4'o_4$ (*14,16*).

**Solution** (*b*):

(Line in *H*, but distance referred to *V*.)

As a point is revolved about a line in *H*, its *H* projection lies always in the *H* trace of the plane of revolution, and the distances from the *H* projections of the moving point to *GL* indicate the distances from the point to *V*.

**Construction** (*b*):

Given: Line *AB* and point *O*, Fig. 5.

Required: To revolve *O* about *AB* into a position $\frac{3}{4}$ inch before *V*.

Since the point is to be $\frac{3}{4}$ inch before *V*, its *H* projection will be $\frac{3}{4}$ inch below *GL*; since it is in plane *R*, its *H* projection is on the *H* trace *HR*. A horizontal line is drawn $\frac{3}{4}$ inch below *GL* (*17,18*) and extended until it intersects the *H* trace at $o_1$, the *H* projection of the required position. A perpendicular to *HR* at $o_1$ (*19*) intersects the circle at $O_{1H}$, the revolved position of the required point. The *V* projection, $o_1'$, is then found above *GL* a distance equal to $o_1O_{1H}$ and on the perpendicular to *GL* through $o_1$ (*20,21*). Another possible position of point *O* would be in the fourth quadrant.

## ROUTINE USE OF REVOLUTION

### PRINCIPLE "D"

**81.** For the very practical reason that men prefer to employ repeatedly an efficient and easily remembered procedure in solving similar problems, the author of this book will deliberately employ the method of rotating planes containing values to be determined about a horizontal line of that plane into the horizontal plane through the line. It is believed this practice has certain definite merits. A horizontal line can easily be assumed in any plane; its elevation or *V* projection is a horizontal line. The horizontal plane through this line may be regarded as the *H* plane of projection. The horizontal projection of the line then becomes the *H* trace of the plane to be revolved, and may be drawn and labeled as a plane trace for purposes of easy identification. The procedure of rotating a plane about its *H* trace into the *H* plane of projection consists of rotating one or more points of the plane about the *H* trace until their paths pierce the horizontal plane, Fig. 6(*a*). If *O* be one point on a line *OM* in

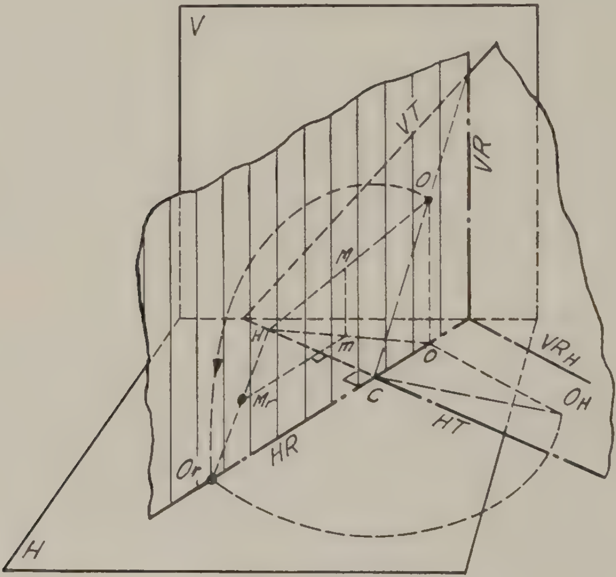


Fig. 6(*a*).

plane $T$, that line extended will pierce $H$ in a point on $HT$. This point $H$ is on the axis of rotation and does not change position; $O_r$ may hence be connected with $H$ to secure the revolved position of $OM$, and $M_r$ secured by erecting a perpendicular to $HT$ through $m$ until it intersects $O_rH$.

While the procedure suggested above can be employed to advantage in most problems, it should not be forgotten that the plane containing angles, plane figures, etc., can be revolved about a line parallel to $V$ until the plane is parallel to $V$ or about a line parallel to $P$ until the plane is parallel to $P$. There are occasions when one of these procedures is preferable.

82. Problem 19. **To find the true value of the angle "$X$" between two intersecting lines.**

**Analysis:**

The process of measuring an angle is comparative, requiring the use of a protractor.

The process of measurement can be accomplished only in some actual working plane, e.g., that of the drawing board.

If the plane of an angle is rotated about any horizontal line in it until it is horizontal, the relation of the lines to each other, i.e., the angle between them, does not change.

**Solution:**

If the plane of the two intersecting lines is revolved about any horizontal line in it until it is horizontal, the revolved angle will show in its true size, Fig. 6($b$).

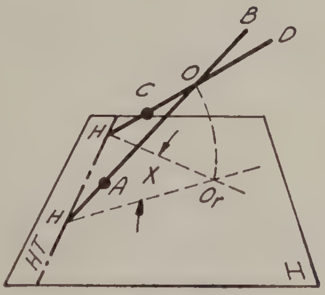


Fig. 6($b$).

**Construction:**

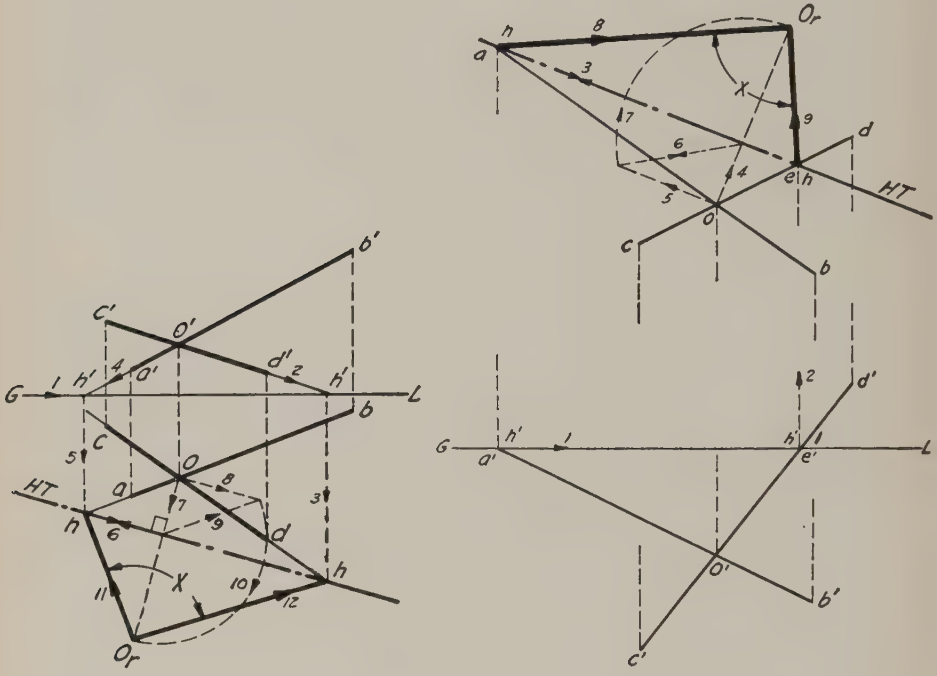
Given: Intersecting lines $AB$ and $CD$, Figs. 6($c$) and 6($d$).

Required: Angle "$X$."

Plane $T$ of $AD$ and $CD$, Fig. 6($c$), intersects $H$ in the line $HT$ ($1$,$6$). Any horizontal line $AE$ is chosen in the plane of lines $AB$ and $CD$, Fig. 6($d$), and the $H$ plane is passed through it ($1$,$3$).

Plane $T$ is revolved about $HT$ into $H$ ($4$,$7$) by the process of rotating point $O$ about $HT$ until its path pierces $H$ at $O_r$.

Points *H* and *H* of *AB* and *CD*, being on the axis of revolution *HT*, do not change position in the process of revolution. The lines *AB* and *CD* hence revolve into the positions $O_rh$ and $O_rh$ (*8*,*9*), [Article 81]. The true value of angle "*X*" then is $hO_rh$ because plane *T* of the angle has been revolved into coincidence with a plane of projection.



(*c*) (*d*)

Fig. 6.

**83.** Problem 19: Example 1. **To find the projections of the bisector of the angle "*X*" between two intersecting lines.**

**Discussion:**

This problem illustrates the very general problem of finding the projections of a point, line, or figure, lying in a plane, when its exact relative position is known. The construction is given in two forms, each of which is applicable under varying circumstances and both of which are equally important.

**Analysis:**

A line is determined by any two points on it.

The bisectors of the projections of an angle are not in general the projections of the bisector of the angle.

An angle can be bisected (graphically) only in its true value. The plane of the angle must therefore be revolved about any horizontal line in it into a horizontal position [Article 82].

Points on an axis of revolution do not change position.

The vertex of the angle and the intersection of the bisector with the axis of revolution will determine the bisector of the angle.

If the bisector does not intersect the axis of revolution in a convenient position an auxiliary line which does intersect the axis as well as the bisector and one side of the angle may be employed, determining first the projections of the auxiliary line and from them the projections of the bisector.

**Solution:**

If the plane of the given lines, Fig. 7(*a*), is revolved about any horizontal line in it into the horizontal plane through the line, the angle "*X*" will show in its true value and may be bisected. If any point of this bisector is revolved back into its true position, a line through this point and the vertex will be the required bisector.

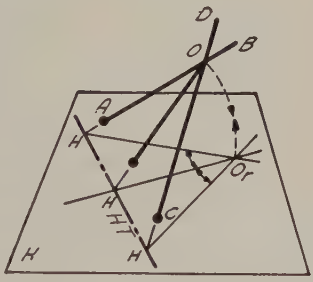


FIG. 7(*a*).

**Construction** (*a*):

Given: Lines *AB* and *CD*, Figs. 7(*b*) and 7(*c*).

Required: Bisector of angle "*X*."

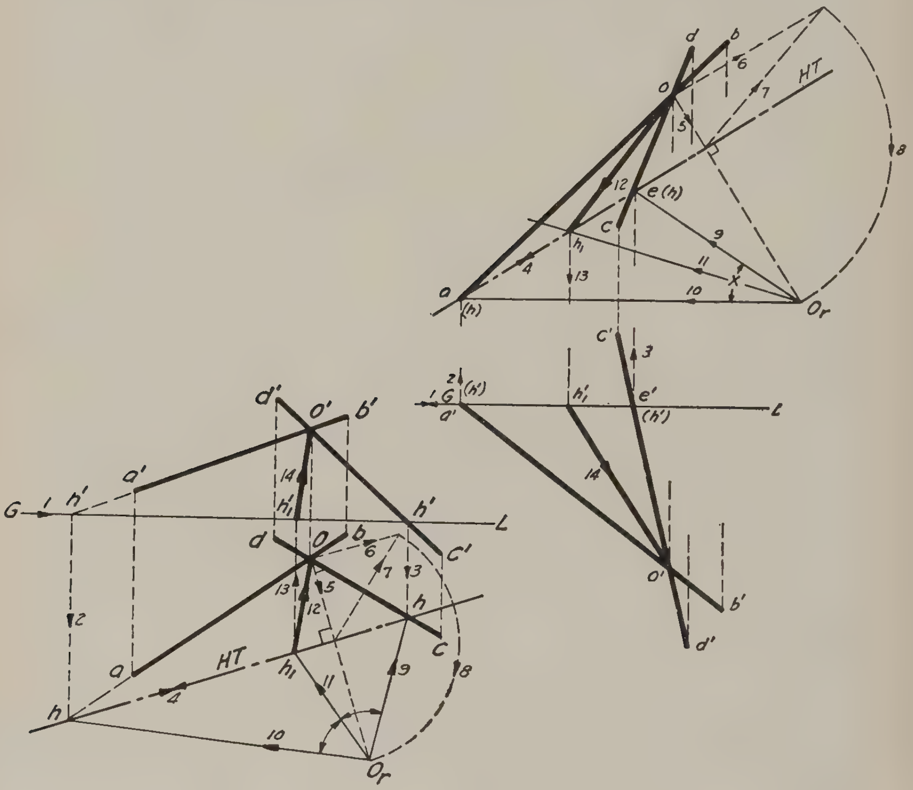
Plane *T* of lines *AB* and *CD* intersects the horizontal plane through *GL* in the horizontal line *HT* (*1,4*), Figs. 7(*b*) and 7(*c*). When plane *T* is revolved about *HT* into this horizontal plane, point *O* falls at $O_r$ (*5,8*). The true value of angle "*X*" is $hO_rh$ and the revolved position of the bisector is $O_rh_1$ (*9,11*). Point $H_1$ is on the axis of revolution and remains unchanged in the revolution. The projections of the bisector are therefore $h_1o$ and $h_1'o'$ (*12,14*).

**Construction** (*b*):

Given: Lines *AB* and *BC*, Fig. 7(*d*).

Required: Bisector of angle "*X*."

Plane *T* of *AB* and *BC* intersects the horizontal plane through *GL* in line *HT* (*1,8*). When the plane is revolved about *HT* into *H*, $A_rB_rC_r$ is the true value of angle "X" (*9,14*), and the revolved position of the bisector is $B_rD_r$ (*15*). To secure the projections of *BD*, the revolved position of any line *AC* in plane *T* (*16*) is drawn intersecting the revolved bisector at $M_r$. The *V* and *H* projections of *AC* are also drawn (*17,18*). A line through $M_r$ perpendicular to *HT* intersects *AC* at *m*



(*b*) (*c*)

Fig. 7.

(19). $m'$ is found on $a'c'$ (20). In this construction the line $M_rm$ constitutes the $H$ trace of the vertical plane in which $M$ revolves about $HT$. The projections of the bisector are then $b'm'$ and $bm$ (21,22).

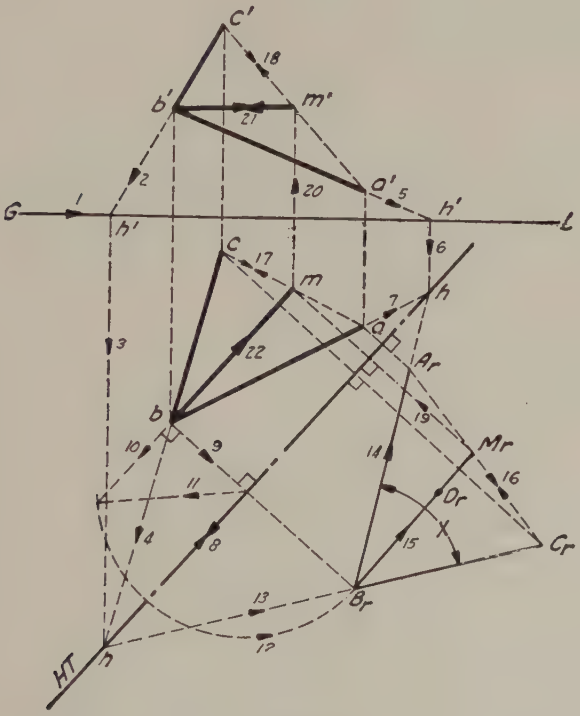


Fig. 7(*d*).

**84.** Problem 19: Example 2. **To determine the true size and shape of a given plane figure.**

**Analysis:** See Analysis of Problem 19.

**Solution:**

If the plane of the figure is revolved about any horizontal line in it until it is horizontal, the figure will show in its true shape and size.

**Construction:**

Given: Triangle *MNS* in the plane *AOB*, Fig. 8.

Required: The true size of the triangle *MNS*.

Any horizontal line *AE* is chosen in plane *AOB* (*1*,*3*). The horizontal plane through *AE* is indicated by *GL* (*1*). Point *M* of the triangle is revolved into the horizontal plane about *HT* to the position $M_r$ (*4*,*7*). Since the two points $H_1$ and $H_2$ in which *MS* and *MN* intersect *HT* remain unchanged in the revolution, $M_rN_r$ and $M_rS_r$ may be drawn from $M_r$ through the points $h_1$ and $h_2$ to secure $N_r$ and $S_r$ (*8*,*11*). $M_rS_rN_r$ is the revolved position and true size of the triangle.

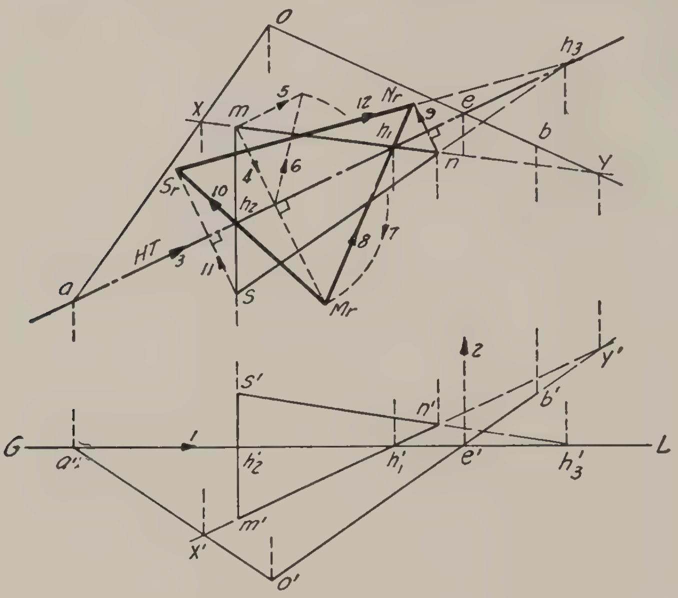


FIG. 8.

85. Problem 19: Example 3. **To determine the projections of a specified plane figure in a specified position in a given plane.**

**Analysis:** See Analysis of Problem 19.

**Solution:**

If the given plane is revolved about any horizontal line in it until it is horizontal, the specified pattern may be constructed in true shape and size in the specified position. The plane and pattern may then be counter-revolved to the given position.

**Construction:**

Given: Plane *AOB*, Fig. 9.

Required: The projections of a square lying in plane *AOB*, sides one inch, one side parallel to *AO* and $\frac{1}{2}$ inch from *AO*. The nearest corner is to be one inch from *O*.

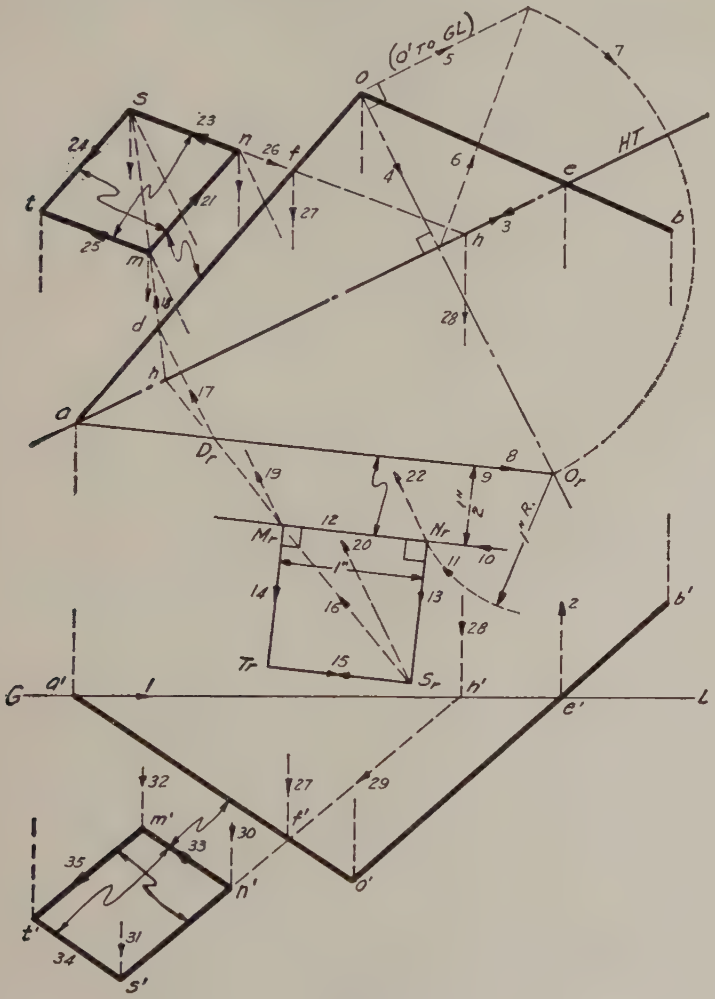


FIG. 9.

Any horizontal line *AE* (*1,3*) is chosen in plane *AOB*. The horizontal plane through *AE* is indicated by *GL*. The intersection of *AOB* with this plane is then *AE* or *HT*. Point *O* is revolved about *HT* into *H* at $O_r$ *(4,7)*. $O_ra$ is the revolved position of line *OA* *(8)*. $M_rN_rS_rT_r$ is the revolved position of the square *(9,15)*. The diagonal of the square $S_rM_r$ *(16,)* intersects $O_ra$ at $D_r$ and *ae* at *h*. Its *H* projection is then *hd* *(17,18)* and the *H* projections of *M* and *S* are at *m* and *s* *(19,20)*. *MN* is given parallel to *AO*, hence *mn* is parallel to *ao* *(21,22)*. The remaining corner, *t*, is determined by lines through *m* and *s* parallel to *sn* and *mn* *(24,25)*.

The *V* projection of the square is obtained by extending one of the sides of the *H* projection to intersect two of the lines of the plane *AO* and *AE* at *F* and *H* *(26,28)*. The *V* projection of *SN* is thus $s'n'$ *(29,31)*. $m'$ is on a parallel to $a'o'$ through $n'$ *(32,33)*. $t'$ is the intersection of parallels to $s'n'$ and $n'm'$ *(34,35)*.

86. Problem 20. **To pass a line through a given point making a given angle with a given line.**

### Analysis:

The process of measuring or constructing an angle is comparative and must be conducted in a working plane.

The plane of the given point and the given line must hence be made to coincide with a working plane.

The process of making one plane coincide with another is that of revolution about their line of intersection, Principle D.

### Solution:

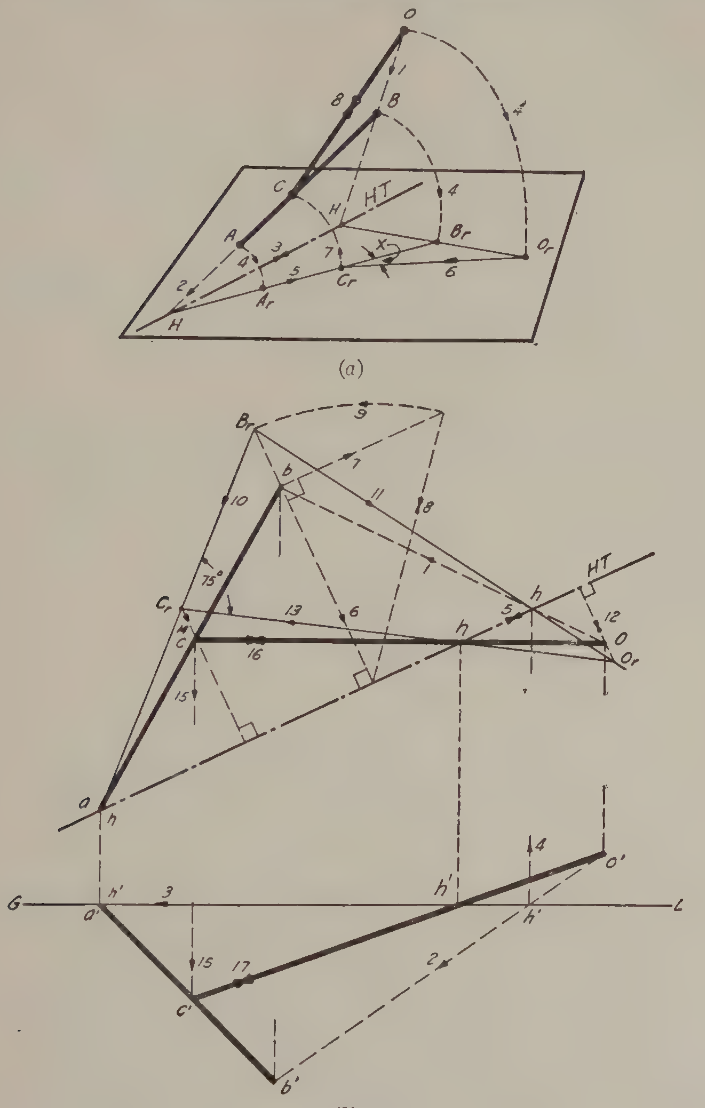
If the plane of the given point and given line is revolved about a convenient horizontal line of the plane into the horizontal plane through this line, the specified angle may be constructed, the vertex determined, and counter-revolved to its true position, Fig. 10(*a*).

### Construction:

Given: Point *O* and line *AB*, Fig. 10(*b*).

Required: The projections of a line through *O* making an angle of 75 degrees with *AB*.

The intersection of the plane *T* of *AOB* with a horizontal plane through *A* is *HT* *(1,5)*. Points *B* and *O* fall at $B_r$ and $O_r$, when *OAB* is revolved about *HT* into *H* *(6,12)*. Line $O_rC_r$ is drawn, making the specified angle of 75 degrees with $aB_r$ (*13*). When point $C$ is counter-revolved from position $C_r$, its projections



(*a*)

(*b*)

FIG. 10.

are found at $c$ and $c'$ (*14,15*) and the projections of the required line are $oc$ and $o'c'$ (*16,17*).

87. Problem 21. **To revolve a given point about a given horizontal line through a given angle, "*X*," and find the projections of the point in this position.**

**Analysis:**

The path of rotation is a circle.

The plane $R$ of the circular path is perpendicular to the given line $AB$.

The center $C$ of the circle is the point in which $AB$ pierces plane $R$.

The radius of the circle is the distance from given point $O$ to $C$.

The specified angle of revolution is measured between the initial radius $CO$ and another radius $CO_1$.

**Solution:**

The point revolves about the line in a circle whose radius equals the distance from the point to the given line, and whose plane is perpendicular to the given line. The angle "$X$" is the angle included between the radius through the given point and the radius through the required point of the circle. The projections of this second point of the circle are the projections of the required revolved position of $O$.

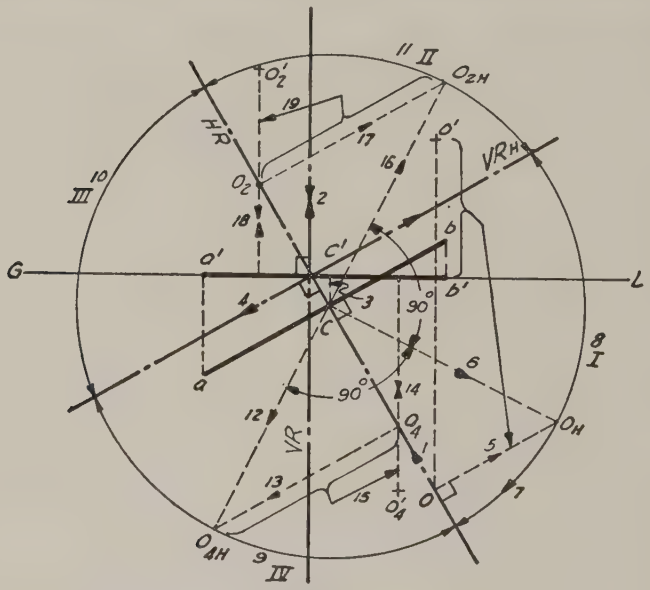


FIG. 11.

**Construction:**

Given: Horizontal line $AB$ and point $O$, Fig. 11.

Required: To revolve $O$ about $AB$ through 90 degrees.

A horizontal plane is passed through $AB$ and it is indicated by extending $a'b'$ as a ground line.

Plane $R$ is passed through point $O$, perpendicular to $AB$ (*1,2*). It intersects $AB$ at point $C$ (*3*). Plane $R$ is then revolved about its $H$ trace $HR$ into $H$ (*4,7*). The arc (*8*) on which $O_H$ is located is in the first quadrant because point $O$ is given in that quadrant. The remaining arcs are in quadrants as shown (*9,11*).

Point $O$ is moved through the specified angle of 90 degrees from $O_H$ to $O_{4H}$ (*12*). This position is designated $O_{4H}$ because it is in the part of the circle which passed through the fourth quadrant. The $H$ and $V$ projections of $O_4$ are then $o_4', o_4$ (*13,15*). Another possible position for point $O$ is $O_2$ in the second quadrant (*16*). The $H$ and $V$ projections of the point in this position are $o_2', o_2$ (*17,19*).

88. Problem 21: Example 1. **To revolve a given point about a given line which is oblique to *H*, *V*, and *P*, through a given angle "*X*" and find the projections of the point in this position.**

**Analysis:**

See Analysis of Problem 21.

**Solution:**

See Solution of Problem 21.

**Construction:**

Given: Line $AB$ oblique to $H$ and $V$, and point $O$, Fig. 12.

Required: To revolve $O$ about $AB$ through 90 degrees.

A ground line (*1*) is assumed wherever desired. Plane $R$ is passed through point $O$, perpendicular to line $AB$ (*2,6*), [Article 56]. $AB$ pierces plane $R$ in point $C$ (*7,11*), [Article 62]. Plane $R$ is revolved about its $H$ trace $HR$ into $H$, by the process of rotating points $O$ and $C$ to positions $O_r$ and $C_r$ (*12,17*), giving the true relative positions of point $O$ and center $C$ in plane geometry. The circle of revolution in its revolved position in the $H$ plane is drawn with $C_r$ as a center and radius $C_rO_r$ (*18*).

Radii $C_rO_{1r}$ and $C_rO_{2r}$, making 90 degrees with $O_rC_r$, give the required revolved positions of points $O_1$ and $O_2$ (*19*). The diameter $O_{1r}$, $O_{2r}$ is extended to intersect *HR* at *H*. The *H* projection of the diameter $O_{1r}O_{2r}$ is then drawn from *h* through *c* (*20*). The *H* projections of $O_1$ and $O_2$ are then $o_1$ and $o_2$ on *hc* (*21* and *26*). The *V* projections are $o_1'$ (*22,25*) and $o_2'$ (*26,30*).

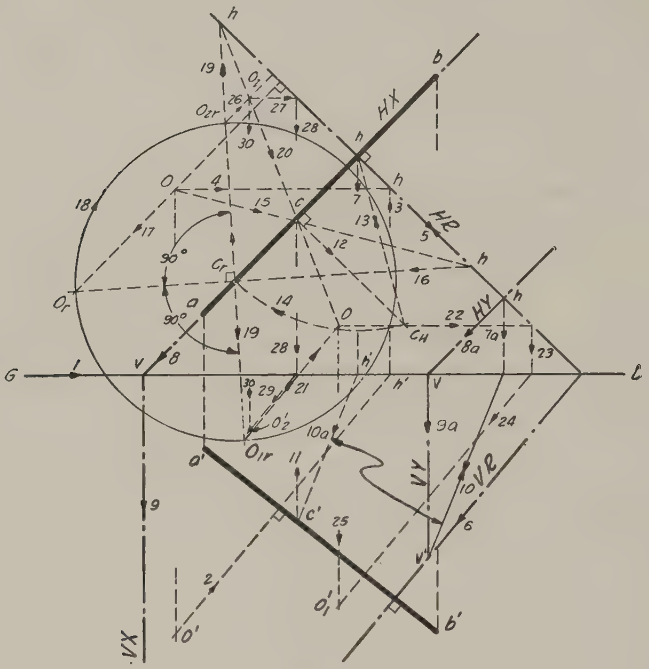


FIG. 12.

89. Problem 22. **To find the angle "*X*" which a given line makes with a given plane.**

### Analysis (*a*), Fig. 13(*a*).

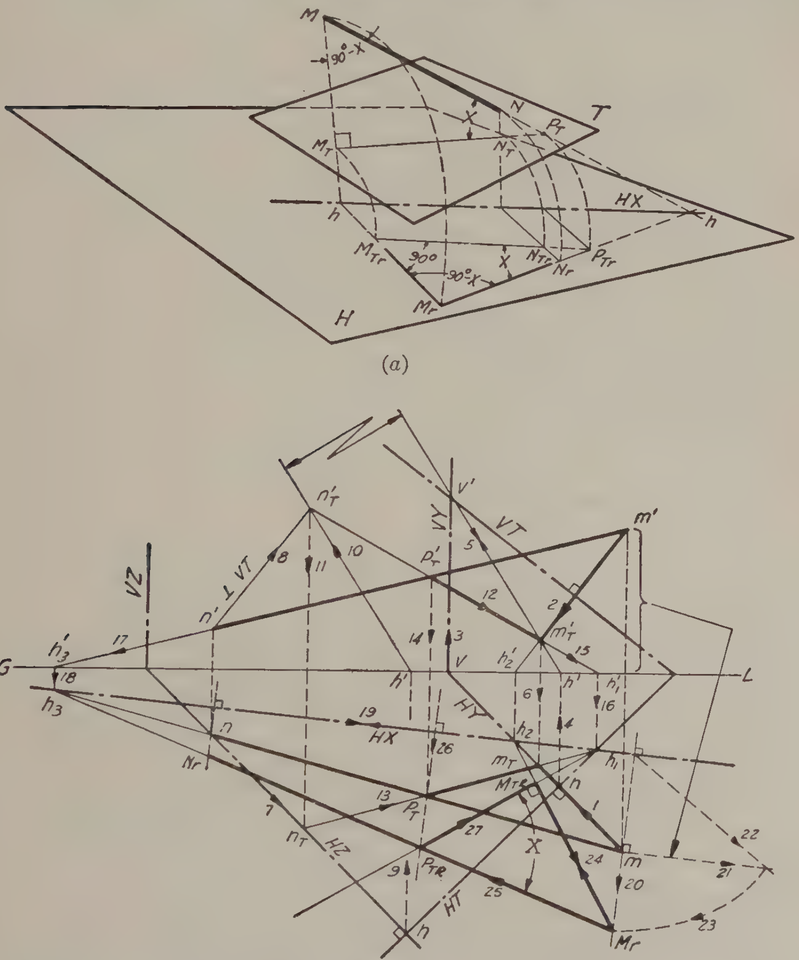
The angle which a line makes with a plane is measured between the line and its projection on the plane.

The projection of the line on the plane is determined by the piercing point of the line in the plane and the piercing point of a perpendicular to the plane from some point of the line.

The value of the angle between the line and its projection will be determined by revolving their plane about some horizontal line of it into coincidence with the *H* plane through that line.

**Solution** (*a*):

If the projection of the given line on the given plane is determined, Fig. 13(*a*), and the plane of these two lines is revolved about any hori-



(*a*)

(*b*)

Fig. 13.

zontal line in it into the horizontal plane of this line, the true value of angle "*X*" is determined.

**Construction** (*a*–1):

Given: Plane *T* and line *MN*, Fig. 13(*b*).

Required: The angle "*X*" which *MN* makes with plane *T*.

The projection of point *M* on *T* is $M_T$ (*1,6*). The projection of point *N* on *T* is $N_T$ (*7,11*). The projection of *MN* on *T* is hence $M_TN_T$ (*12,13*) and the piercing point of *MN* in *T* is $P_T$ (*14*).

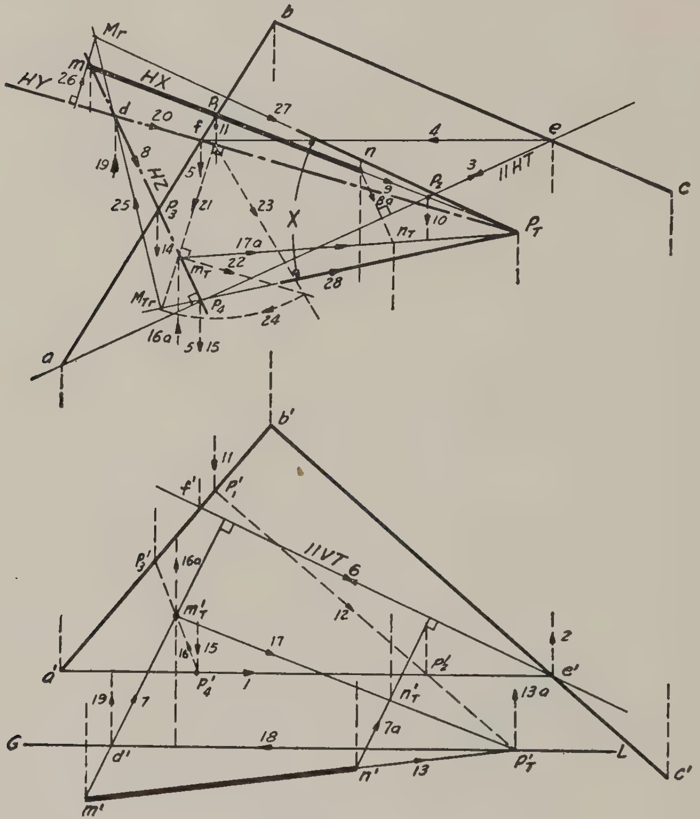


FIG. 13(*c*).

The *H* trace of plane *X* of *MN* and its projection on *T* is *HX* (*15,19*). When point *M* is rotated about *HX* into *H* it falls at $M_r$ (*20,23*). $MH_2$ and $MH_3$ show in their true relative positions as $M_rh_2$ and $M_rh_3$ (*24,25*). $P_T$ is found at $P_{T_r}$ (*26*) and $P_TM_T$ at $P_{T_r}M_{T_r}$ (*27*). The true value of angle *X* is thus $M_rP_{T_r}M_{T_r}$ (*lines 25 and 27*).

**Construction** (*a*–2):

Given: Plane *ABC* and line *MN*, Fig. 13(*c*).

Required: The angle "*X*" which *MN* makes with the plane *T* (*ABC*).

The directions of the *H* and *V* traces of plane *T* (*ABC*) are *ae* (*1,3*) and *f′e′* (*4,6*). The projection of *MN* on *T* is determined by passing a perpendicular to *T* through *M* (*7,8*) and by finding the points where *MN* and the perpendicular from *M* pierce *T*. The piercing point of *MN* in *T* is $P_T$, using *H* projecting plane *X* (*9,13*), [Article 62]. The piercing point of the perpendicular to *T* through *M* is $M_T$, using *H* projecting plane *Z* (*14,16a*). The projection of *N* on *T* is $N_T$ (*7a,8a*) and the projection of *MN* on *T* is $m_T n_T$, $m'_T n'_T$ (*17,17a*). The horizontal line $P_T D$ through $P_T$ in plane $MP_T M_T$ (*18,20*) is used as the axis of revolution to determine the value of angle "*X*." $dp_T$ or *HY* (*20*) is the horizontal trace of this plane. Point $M_T$ falls at $M_{T_r}$ when revolved into *H* about *HY* (*21,24*), and *M* at $M_r$ (*25,26*). Point $P_T$, the vertex of angle "*X*," is on the axis of revolution, hence does not change position. The true value of angle "*X*" which *MN* makes with *T* is thus $M_r P_T M_{T_r}$ (*27,28*).

**Analysis** (*b*):

The given line, the perpendicular from one point of it to the plane, and the projection of the line on the plane form a right triangle in which the angle between the line and the perpendicular is the complement of the required angle.

The value of the angle may thus be determined by rotating the plane of the given line and a perpendicular to the plane into a convenient horizontal plane without having determined the projection of the line on the plane.

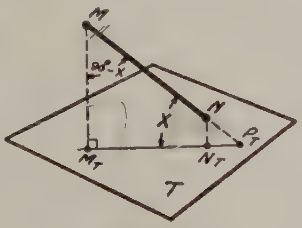


FIG. 13(*d*).

**Solution** (*b*):

If through any point of the given line a perpendicular to the given plane is passed, Fig. 13(*d*), and the plane of these lines is revolved about any horizontal line in it into the horizontal plane through it, the value of the angle is determined. Its complement is the value of the required angle "*X*."

**Construction** (*b*):

Given: Plane *ABC* and line *MN*, Fig. 13(*e*).

Required: The angle which *MN* makes with the plane *T* (*ABC*).

The directions of the *H* and *V* traces of plane *T* (*ABC*) are *ce* (*1,3*) and *a′f′* (*4,6*). A perpendicular to plane *T* is passed through point *M* (*7,8*).

Any horizontal line *KS* is chosen in the plane of *MN* and the perpendicular through *M* to *ABC* (*9,12*). *HX* (*12*) is the *H* trace of *KMS* in the horizontal plane through *KS*. Point *M* falls at $M_r$ when the plane of *KMS* is revolved about *HX* into *H* (*13,16*). $kM_rs$ is the true value of the complement of the required angle. A perpendicular $M_rO_r$ (*19*) to $kM_r$ is drawn through $M_r$. Angle $sM_rO_r$ is then the true value of the required angle which *MN* makes with plane *ABC*.

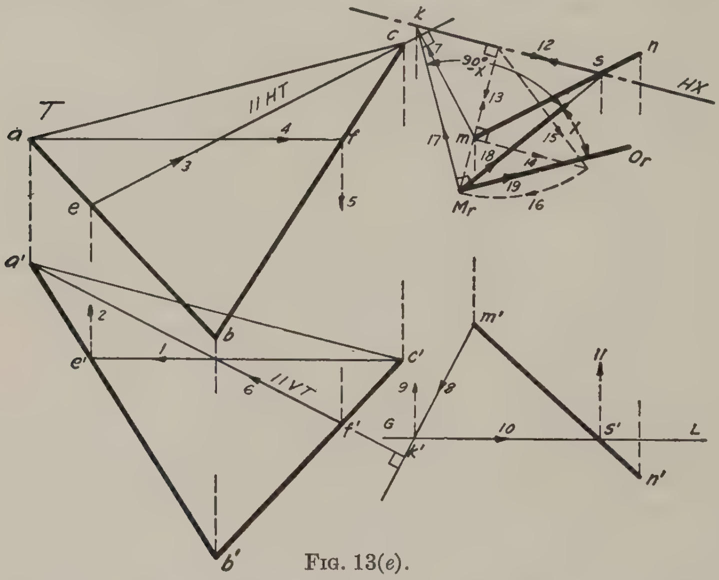


FIG. 13(*e*).

**90.** Problem 22: Example 1. **To find the angles θ and φ which a given line makes with the *H* and *V* planes, respectively.**

**Discussion:**

Angles *θ* and *φ* are the angles which a line makes with its *H* and *V* projections. Methods A and B of finding the true length of a line will therefore obtain these angles.

**Analysis:**

The angles which a line makes with $H$ and $V$ are measured between the line and its $H$ and $V$ projections respectively.

**Solution:**

Method A. If the given line is revolved about a projecting line (e.g., $H$) of any point of it until the line is parallel to $V$, Fig. 14($a$), the angle $\theta$ is then projected upon the $V$ plane in its true value.

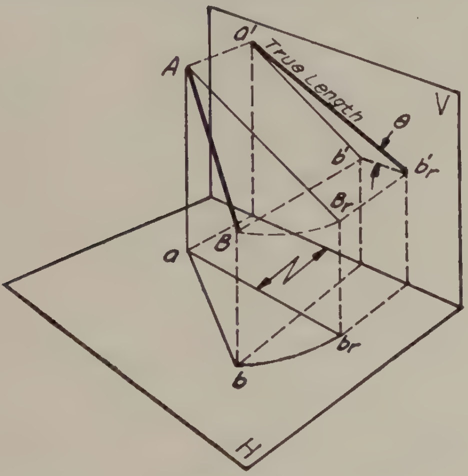


FIG. 14($a$).

Method B. If the line is revolved first about its $H$ projection into $H$, Fig. 14($b$), and about its $V$ projection into $V$, the true value of the angles $\theta$ and $\phi$ will be determined.

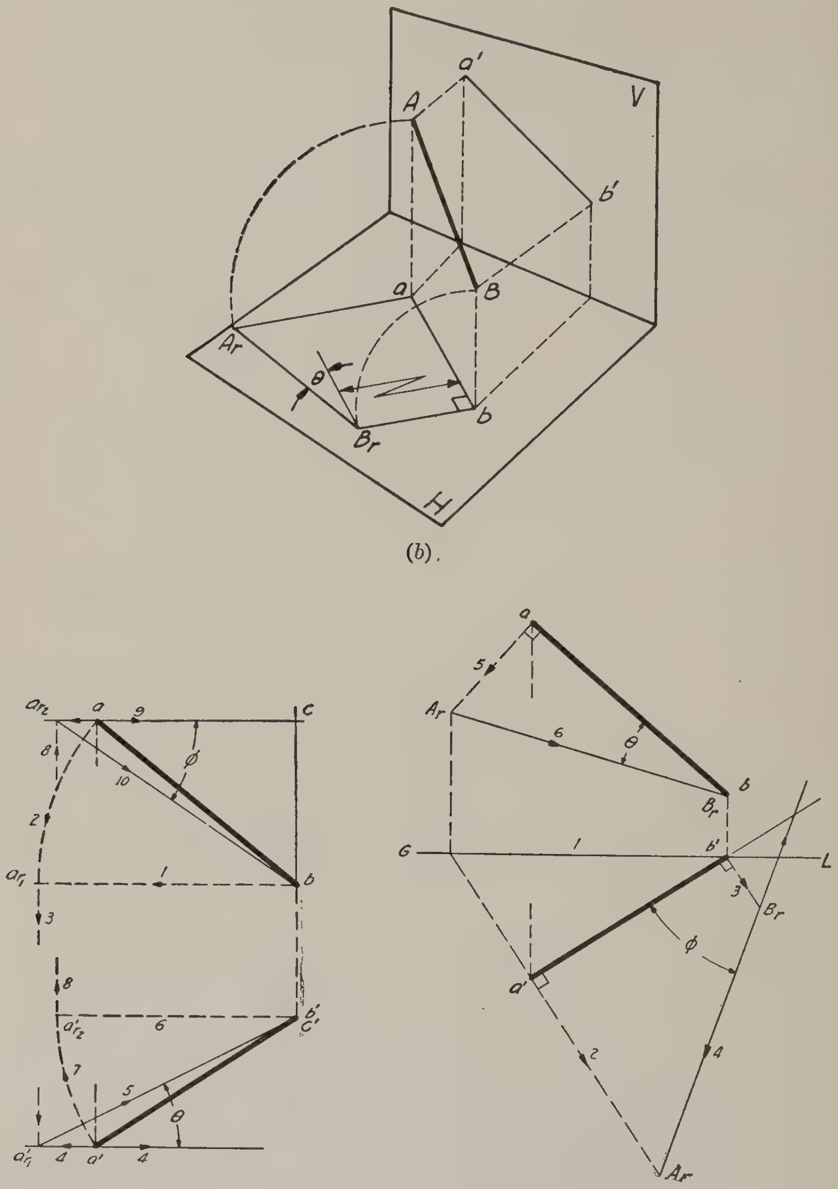
**Construction:**

Given: Line $AB$, Figs. 14($c$) and 14($d$).

Required: The angles $\theta$ and $\phi$, which $AB$ makes with $H$ and $V$.

A. Point $A$ is rotated about the $H$ projecting line of $B$, Fig. 14($c$), until it is the same distance from $V$ as is $B$ (*1,5*). The true value of $\theta$ is then projected upon the $V$ plane as $b'a'_{r1}a'$ (*4,5*). $A$ is then rotated about the $V$ projecting line of $B$ until it is the same distance from $H$ as is $B$ (*6,10*). The true value of angle $\phi$ is then projected upon the $H$ plane as $ba_{r2}a$ (*9,10*).

B. A horizontal plane is assumed through point $B$ (*1*), Fig. 14($d$). $AB$ is then revolved about its $V$ projection $a'b'$ into $V$


a'
A
V
b'
a
B
Ar
θ
b
Br
H
ar2
a
9
c
8
ϕ
10
2
ar1
1
b
3
8
b'
a'r2
6
c'
7
5
θ
a'r1
4
a'
4
a
5
Ar
6
θ
b
Br
G
1
b'
L
3
Br
ϕ
a'
2
4
Ar

(b).

(c)

(d)

Fig. 14.

at $A_rB_r$ (*2,4*). The true value of angle $\phi$ is then shown between $A_rB_r$ (*4*) and the *V* projection $a'b'$. *AB* is revolved about its *H* projection *ab* into *H* (*5,6*). The true value of the angle $\theta$ is then shown between $A_rB_r$ (*6*) and the *H* projection *ab*.

91. Problem 22: Example 2. **To pass a line through a given point making a given angle "*X*" with a given plane.**

**Analysis:** (Converse of Problem 22).

The given angle may be measured between the required line and its projection on the given plane.

The complement of the given angle will be measured between the required line and a perpendicular to the given plane through the given point.

Any plane through the perpendicular from the given point to the given plane may serve as the plane of the angle and required line.

If any such plane is revolved about any horizontal line in it into the horizontal plane through the line, the complementary angle and revolved position of the required line may be constructed and counter-revolved to their true positions.

**Solution:**

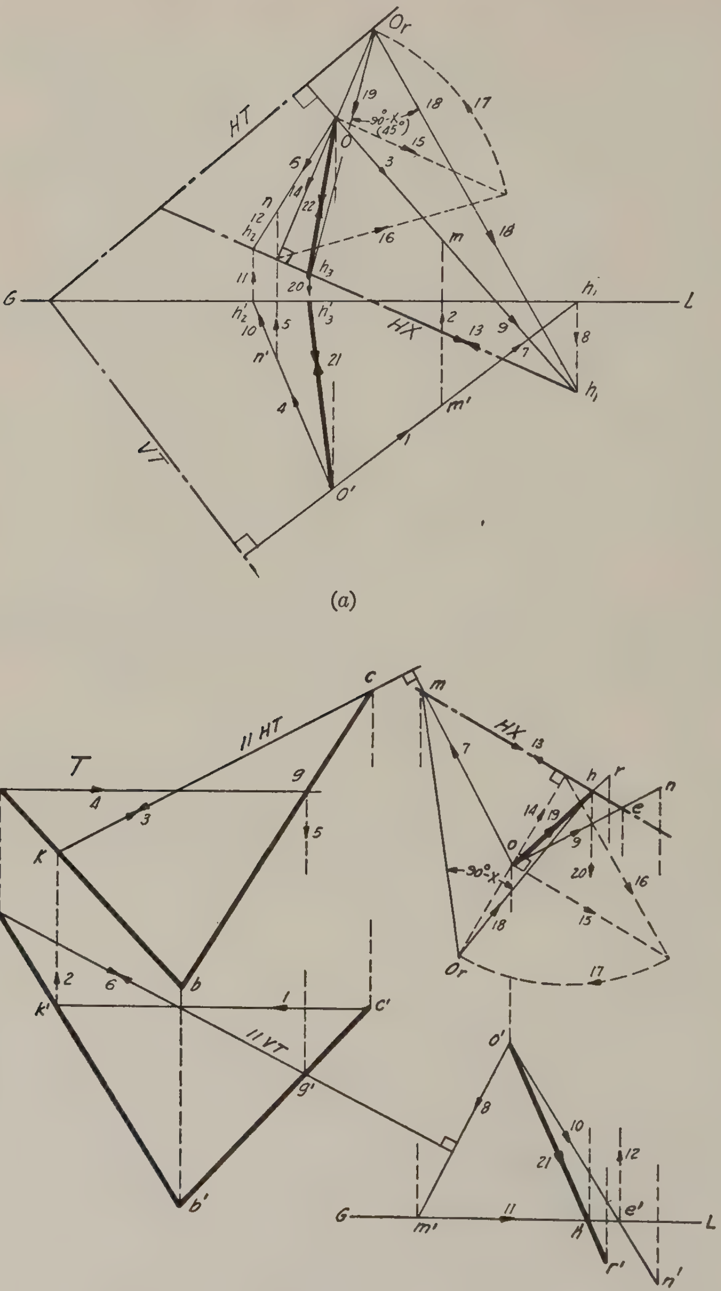
If through the given point a line perpendicular to the given plane is passed, the required line may then be passed through the given point, making with the perpendicular an angle equal to the complement of the given angle [Article 86].

**Construction** (*a*):

Given: Point *O* and plane *T*, Fig. 15(*a*).

Required: A line through *O* making 45 degrees with plane *T*.

*OM* is passed through point *O* perpendicular to plane *T* (*1,3*). Any other line *ON* is passed through *O* (*4,6*). The *H* trace *HX* of plane *NOM* passes through points $H_1$ and $H_2$, (*7,9*) and (*10,12*). When plane *NOM* (*X*) is revolved about *HX* (*13*) into *H*, point *O* falls at $O_r$ (*14,17*), and line *OM* at $O_rh_1$ (*18*). The revolved position of the required line is then drawn at the angle of $90° - X$ (*45°*) with $O_rh_1$ (*19*). It intersects *HX* at $h_3$. The projections of the required line are therefore $o'h_3'$ and $oh_3$ (*21,22*).


Or
HT
19
18
17
90°-X°
(45°)
O
15
3
5
14
22
n
12
$h_2$
16
m
18
$h_3$
11
20
G
$h_1$
L
$h'_2$
$h'_3$
HX
2
13
9
8
10
5
7
$h_1$
n'
21
4
m'
1
VT
O'
c
m
|| HT
HX
13
7
T
a
9
4
3
h
r
n
14
19
e
5
K
O
9
a'
90°-X
20
16
18
15
Or
17
2
6
b
1
C'
k'
|| VT
o'
g'
8
10
12
21
G
11
m'
H
e'
L
r'
n'
b'

(a)

(b)

FIG. 15.

**Construction** (*b*):

Given: Point *O* and plane *T* (*ABC*), Fig. 15(*b*).

Required: A line through *O* making 45 degrees with plane *ABC*.

The directions of the traces of plane *ABC* are *kc* (*1,3*) and *a′g′* (*4,6*). The projections of the perpendicular through *O* to plane *ABC* are hence *om* (*7*) and *o′m′* (*8*). Any other line *ON* is passed through point *O* (*9, 10*). Any horizontal line *ME* is selected in the plane of *MON* (*11,13*). *HX* (*13*) is the *H* trace of plane *MON* in the horizontal plane through *ME*. *O* falls at $O_r$ when revolved into the *H* plane about *HX* (*14,17*). The revolved position of the required line is drawn through $O_r$ at the angle $90° - X$ with $mO_r$ (*18*), intersecting *HX* at *h*. The projections of the required line are *oh* and *o′h′* (*19,21*).

**92.** Problem 23. **To find the angle "*X*" between two intersecting planes.**

**Analysis** (*a*):

The angle between two planes *R* and *S* is measured between two lines, one in either plane and perpendicular to the line of intersection at the same point.

The above two lines define a third plane, *T*, which is perpendicular to each of the given planes and to their line of intersection at the vertex of the angle.

**Solution** (*a*):

If through any chosen point *O* on the line of intersection of the given planes *R* and *S*, Fig. 16(*a*), plane *T* is passed perpendicular to this line, the angle between the lines *OC* and *OD* cut by this plane from the given planes will be the required angle "*X*."

**Construction** (*a*):

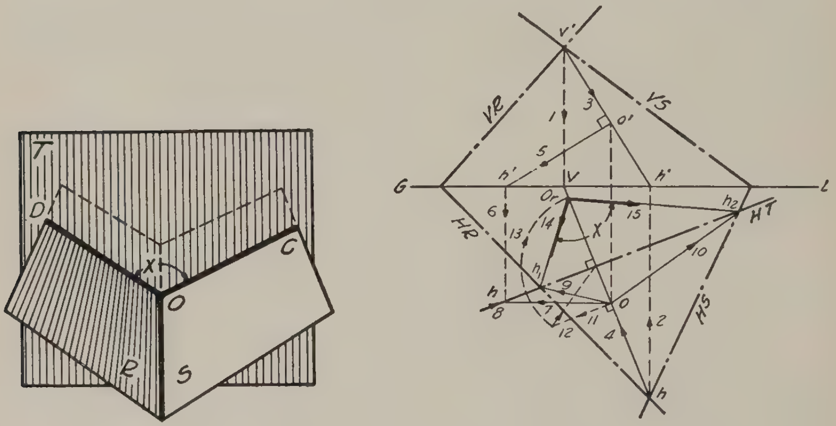
Given: Planes *S* and *R*, Figs. 16(*b*) and 16(*c*). (See Fig. 17[*c*].)

Required: Angle "*X*."

The projections of the line of intersection of planes *S* and *R*, Fig. 16(*b*), are *v′h′* and *vh* (*1,4*). In Fig. 16(*c*), *MN*, the line of intersection of *S* and *R* is given. Plane *T* is passed through any point *O* on the line of intersection *VH*, Fig. 16(*b*), *MN*, Fig. 16(*c*), and perpendicular to it (*5,8*); its *H* trace is *HT* (*8*); the *V* trace is not required and is not drawn. Since the point *O* was chosen on the line of intersection of the two planes, and the plane $T$ passed through $O$, the lines cut from $S$ and $R$ by $T$ meet at $O$; $oh_1$ and $oh_2$ (*9,10*) are then the $H$ projections of the lines cut by $T$ from $S$ and $R$. Plane $T$ is revolved about $HT$ into $H$ (*11,13*), and $h_1O_rh_2$ is the true value of the required angle "$X$" (*14,15*).

**Analysis** (*b*):

Lines drawn from any given point $A$ in space, and perpendicular respectively to two given intersecting planes $R$ and $S$, determine a third plane $T$, perpendicular to both these given planes. The two perpendiculars through $A$, together with the lines cut from $R$ and $S$ by the third plane $T$, form a quadrilateral, two of whose angles are right angles and the remaining two are supplementary.



(*a*) (*b*)

FIG. 16.

It follows, then, that lines passed through any chosen point, and perpendicular respectively to two given intersecting planes, make with each other an angle equal to one of the two angles between the given planes and supplementary to the other.

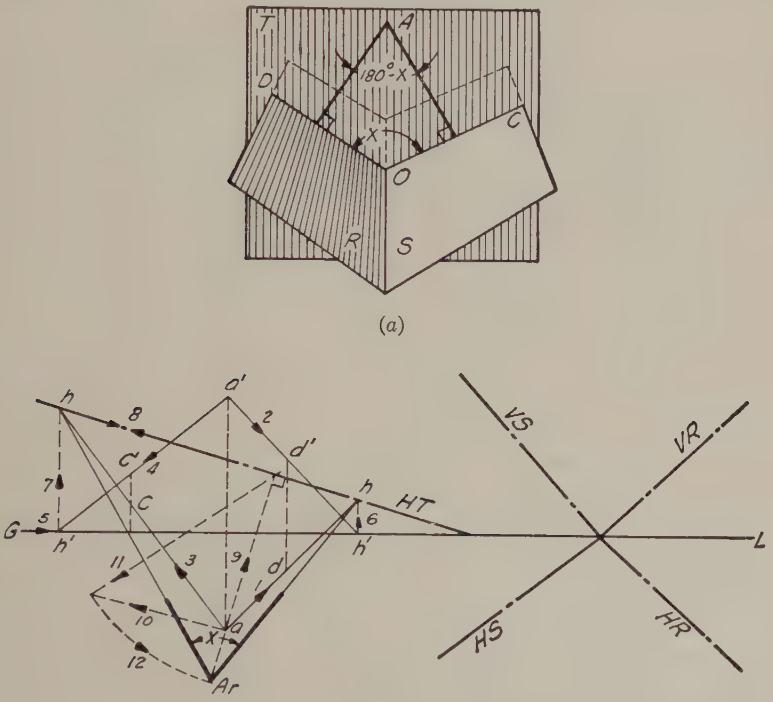
**Solution** (*b*):

If through any chosen point in space two lines perpendicular respectively to the given planes are passed, Fig. 17(*a*), the angle between these lines will be the supplement of the required angle "$X$."

**Construction** (*b*):

Given: Planes *S* and *R*, Figs. 17(*b*) and 17(*c*).
Required: Angle "*X*."

Point *A* is assumed in any convenient position and through it *AC* is passed perpendicular to *S*, and *AD* perpendicular to *R* (*1,4*). *HT* is the *H* trace of the plane of *CAD* (*5,8*). Plane *T* is revolved about *HT* into *H* (*9,12*). $hA_rh$ is the true value of *CAD*, the supplement of the required angle "*X*."



(*a*)

(*b*)

Fig. 17.

93. Problem 23: Example 1. **To find the angle θ, φ, or π of a given plane.**

**Analysis:**

The angle *θ* of a plane *T* is the angle which it makes with any horizontal plane. The angle *φ* is the angle which *T* makes with

*V* or with a plane parallel to *V*. And the angle $\pi$ is the angle which *T* makes with *P* or with a plane parallel to *P*.

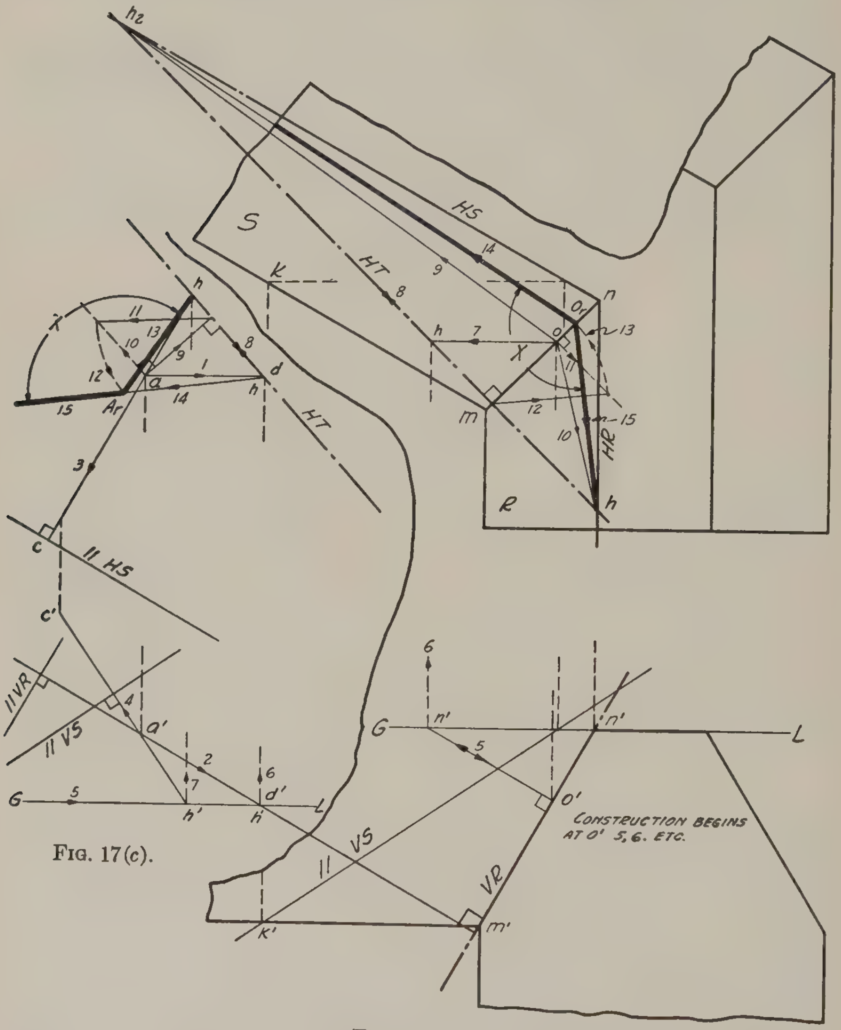


FIG. 17(*c*).

FIG. 16(*c*).

The angle $\theta$ is measured between two lines, one in *T* and one in *H*, and each perpendicular to the intersection of *T* and *H* and the same point.

**Solution:**

If through any point on the $H$ trace of the given plane, Fig. 18($a$), or on the intersection of the given plane and any assumed horizontal plane, Fig. 18($b$), a plane is passed perpendicular to the line, it will cut from the given and horizontal planes two lines which make with each other the required angle $\theta$.

**Construction:**

Given: Planes $T$, Figs. 18($a$) and 18($b$).

Required: Angle $\theta$ of each plane $T$.

Point $O$ is assumed on the given $H$ trace, Fig. 18($a$), and on $HT$ in any horizontal plane, Fig. 18($b$). Planes $X$ through these points intersect $T$ and $H$ in $OF$ and $OV$, Fig. 18($a$), and in $OF$ and $OD$, Fig. 18($b$). The true value of the angle at $O$ ($\theta$) is determined by rotating plane $X$ about the vertical line through $F$ until it is parallel to $V$.

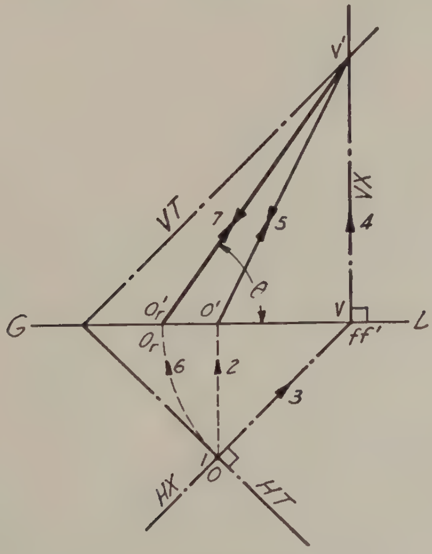


Fig. 18($a$).

94. Problem 23: Example 2. **Through a given line $AC$ in a given plane $R$, to pass plane $S$, making a specified angle "$X$" with plane $R$.**

**Analysis:**

The given line is the intersection of the given and required planes.

The specified angle is measured between two lines both perpendicular to the given line at the same point. One of these lines lies in the given plane and the other in the required plane.

These two lines define a plane perpendicular to the given line at any chosen point.

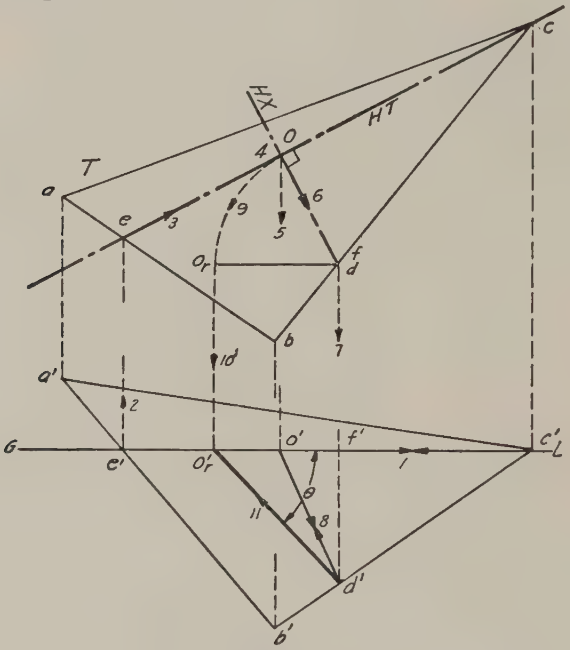


FIG. 18(*b*).

**Solution:**

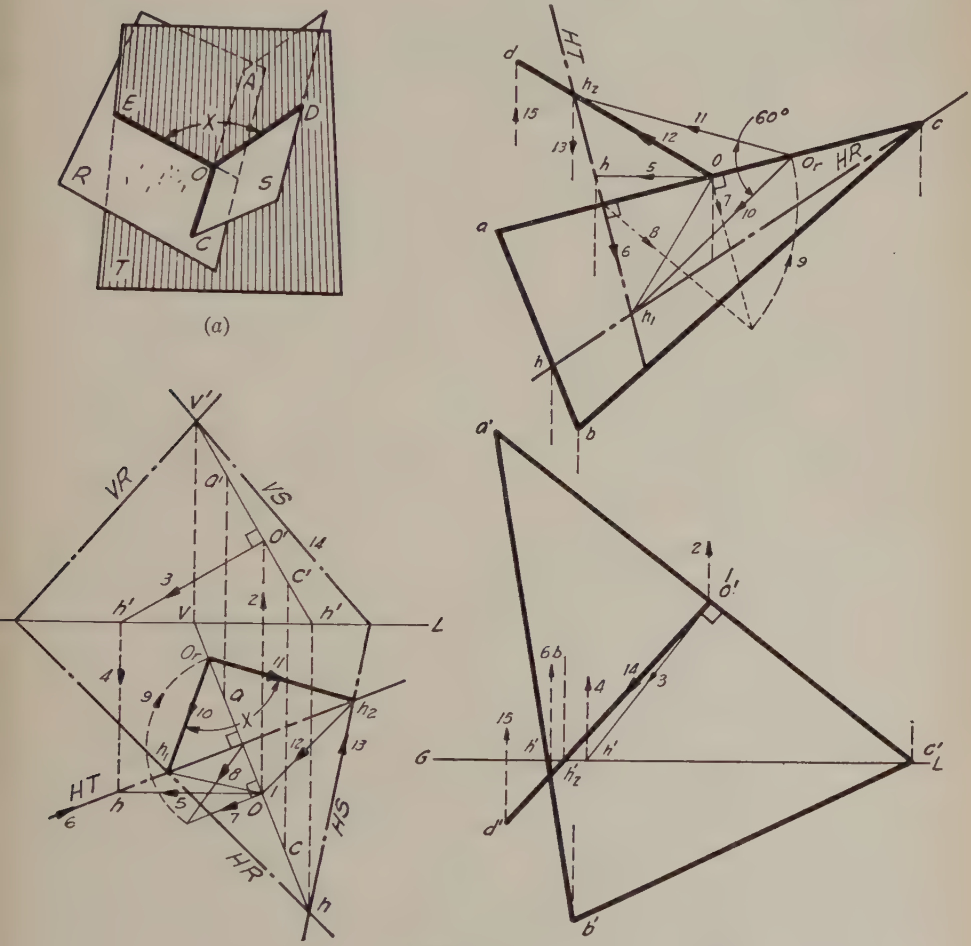
Through any point *O* of the given line *AC* in plane *R*, Fig. 19(*a*), plane *T* is passed perpendicular to *AC*. Line *OE* is the intersection of planes *T* and *R*. Line *OD* is constructed in plane *T* and makes the specified angle "*X*" with line *OE*. This line *OD* and the given line *AC* define the required plane *S*.

**Construction:**

Given: Plane *R* and line *AC* lying in plane *R*, Figs. 19(*b*) and 19(*c*).

Required: Plane through *AC* making 90 degrees with *R*, Fig. 19(*b*); 60 degrees, Fig. 19(*c*).

Through any point $O$ *(1,2)* on $AC$, plane $T$ is passed perpendicular to $AC$ *(3,6)*, [Article 56]; its $H$ trace $HT$ passes through $h$ of the *vertical* and is perpendicular to $ac$. $oh_1$ is the $H$ projection of the line of intersection of $R$ and $T$. After plane $T$ is revolved about $HT$ into $H$ *(7,9)*, $O$ falls at $O_r$ and $h_1O_r$ *(10)* is the revolved position of the intersection of $R$ and $T$. $O_rh_2$ *(11)* is drawn through $O_r$, making 90 degrees with $O_rh_1$ (60 degrees for Fig. 19(*c*)). Since $OH_2$ is a line of plane $T$, its $H$ piercing point lies on $HT$ at the point $h_2$; and $oh_2$ is the $H$ pro-



(*a*)

(*b*) FIG. 19. (*c*)

jection of the required line $OH_2$ of plane $S$, which measures with $OH_1$ of plane $R$ the angle of 90 degrees; (60 degrees, Fig. 19(*c*)). The $H$ trace (*13*) of the required plane, Fig. 19(*b*), then passes through $h_2$ of line $OH_2$ and $h$ of line $AC$. The $V$ trace passes through $v'$ of $AC$ (*14*). It should be understood that $OH_2$ and $AC$ likewise define the required plane $S$. In Fig. 19(*c*) plane $S$ is defined by the given line $AC$ and $OH_2D$. The required plane is therefore $DOC$.

# *PART II*

## CHAPTER VII

## LINES AND SURFACES

### CLASSIFICATION OF LINES

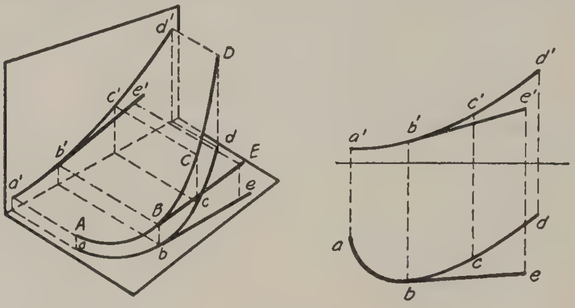
- **Lines**
  - Straight (or Right)
  - Curved
    - Single Curved
      - Conics
        - Circle
        - Ellipse
        - Parabola
        - Hyperbola
      - Spirals
        - Archimedean
        - Logarithmic
        - Hyperbolic
        - Involute of a Circle
      - Trigonometric
        - Sinusoid
        - Tangent
        - Secant
      - Logarithmic
      - Trochoids
        - Cycloid
        - Epicycloid
        - Hypocycloid
    - Double Curved
      - Helices
        - Cylindrical
        - Conical
        - Spherical

### LINES

**95.** A straight line has been defined in Article 23 as generated by the motion of a point in a fixed direction. A curved line is generated by a point moving in a constantly changing direction. A **line of single curvature,** commonly called a **plane curve,** is a line all points of which lie in a plane. A **line of double curvature,** commonly called a **space curve,** is a line only three successive points of which lie in the same plane.

## PROJECTIONS OF CURVED LINES AND TANGENTS

**96.** *The projections of a curve are composed of the projections of the component points of the curve.* If a sufficient number of the points of a curve are projected upon the *H* plane and a smooth curve is drawn through these projections, the resultant curve is the *H* projection of the given curve. (Compare Article 23.) If through the *V* projections of the same or other points of a curve a smooth curve is drawn, this curve is the *V* projection of the given curve, Fig. 1(*a*).



(*a*) (*b*)

FIG. 1:

**97.** If the two projections of a curve are given, the curve is in general completely determined. For if from some point in one projection of a curve a perpendicular to *GL* is drawn until it intersects the other projection of the curve, these two points are projections of a definite point in space, Fig. 1(*b*).

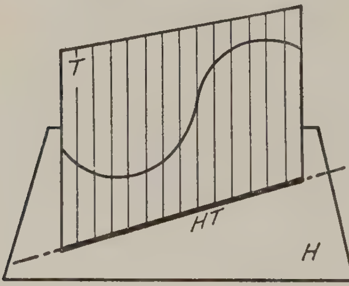


FIG. 2.

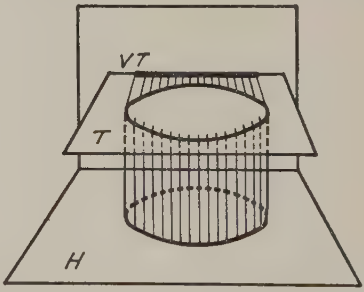


FIG. 3.

**98.** If the plane of a curve of single curvature is perpendicular to *H*, Fig. 2, the projection of the curve on that plane is a straight line coinciding with the $H$ trace of the plane. In like manner the $V$ projection of a curve may be a straight line.

If the plane of the curve is perpendicular to $GL$, both the $H$ and $V$ projections of the curve are straight lines and perpendicular to $GL$.

If the plane of a curve is parallel to either $H$ or $V$, Fig. 3, its projection on that plane is an identical curve, the other projection a straight line parallel to $GL$.

## TANGENTS AND NORMALS TO CURVES OF SINGLE CURVATURE

**99.** A line which lies in the plane of a given curve and has two consecutive points in common with it is called a **tangent** to that curve, Fig. 1($a$). A line lying in the plane of the curve and perpendicular to the tangent at one of these consecutive points of tangency is a **normal** of the curve.

If two curves lie in the same plane and have two consecutive points in common, they are called **tangent curves.**

**100.** *If a straight line and a curve or two curved lines are tangent in space, their projections are tangent.* For if $H$ projecting lines are passed through the several points of the two curves, the lines from the two common consecutive points coincide; the $H$ projections of the curves therefore have two points in common and these points consecutive; the $H$ projections of the given curves are hence tangent. The $V$ projections are likewise tangent, and the *projections of the consecutive points of tangency are the points of tangency of the projections*, Fig. 1($b$).

## SURFACES

**101.** Surfaces in general are conceived as being generated by a line moving according to some law. They are called **ruled surfaces** if they can be generated by a *straight line*, and **double curved surfaces** if the *moving line* cannot be other than a *curve.*

The *moving line* which generates the surface is called the **generatrix.** Any position of the generatrix is called an **element** of the surface.

## CLASSIFICATION OF SURFACES

- Surfaces
  - Ruled
    - Plane
    - Single Curved
      - Cylinders
      - Cones
      - Convolutes
    - Warped
      - Hyperbolic Paraboloid
      - Hyperboloid of Revolution of One Nappe
      - Helicoid
      - Conoid
      - General Forms
        - 3 Linear Directrices
        - 2 Linear Dir. and a Plane Dir.
  - Double Curved
    - Sphere
    - Ellipsoid
      - Spheroid (Ellipsoid of Revolution)
        - Prolate
        - Oblate
      - Two Axes Equal
      - General Form
      - Three Axes Unequal
    - Paraboloid
      - Paraboloid of Revolution
      - Elliptical Paraboloid
    - Hyperboloid of Two Nappes
      - Hyperboloid of Revolution of Two Nappes
      - Elliptical Hyperboloid
    - Torus

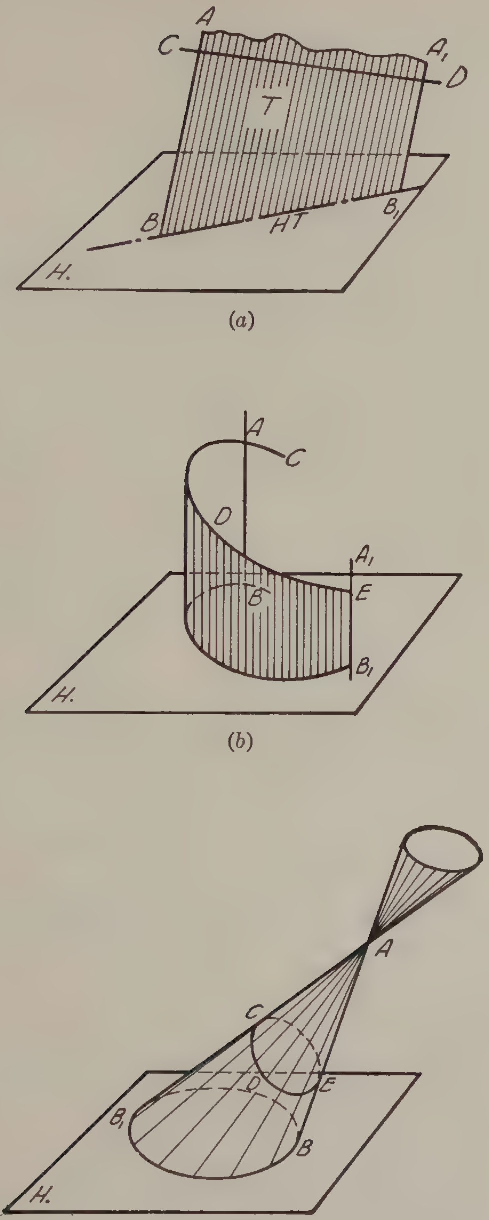
## RULED SURFACES

**102. Ruled surfaces** are generated by a straight line moving according to some law.

*a.* **Planes** are generated by a straight line moving so as to touch another straight line and having all of its positions parallel, Fig. 4(*a*).

*b.* **Single curved surfaces,** Figs. 4(*b*) and 4(*c*), are generated by a straight line, moving so that it constantly touches a given curve. Single curved surfaces are divided into three classes, which are:

**Cylinders.** All positions of the generating line *parallel.*



(a)

(b)

(c)

FIG. 4.

**Cones.** All positions of the generating line *intersecting in a common point.*

**Convolutes.** All positions of the generating line *tangent to a given space curve.*

## CYLINDERS

**103.** A cylinder may be generated by a straight line moving so that it is always parallel to its first position and constantly touches a given curve. If the *cylinder is intersected by a plane,* the *curve of intersection* may be considered the **base** of the cylinder.

If we wish to consider a definite portion of the surface, Fig. 5, a second plane may be passed parallel to the first, intersecting the cylinder in a second base, one being called the **lower base,** the other the **upper base.** According to the direction of the axis, such bases might also be "**front**" and "**rear**" or "**right**" and "**left.**"

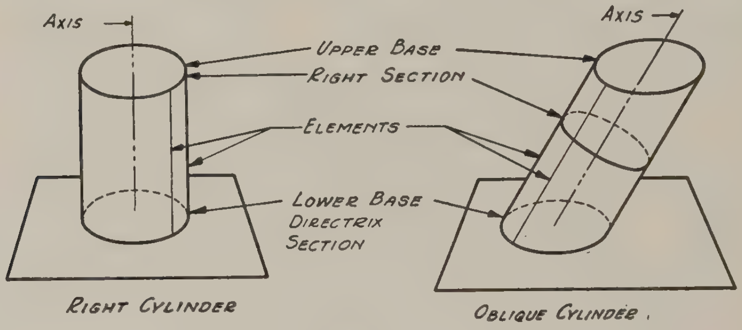


Fig. 5.

The portion of the surface between these two bases is termed "**the cylinder.**"

A line through the center of the base and parallel to the elements is termed **the axis** of the cylinder.

A cylinder whose *elements* are *perpendicular* to the *plane of the base* is termed a **right cylinder.**

A cylinder whose *elements* are *oblique* to the *plane of the base* is termed an **oblique** cylinder.

A *plane perpendicular to the elements* of a cylinder intersects it in a **right section.**

A *cylinder* is *represented* in general by the *projection of a base,* its *axis* and the *limiting elements* in each projection.

The *shape of the base* and the *relative position of the axis* of a cylinder are usually indicated by designating it as a **right-circular cylinder, oblique-elliptical,** etc.

## PROJECTING CYLINDERS

The ***H* or *V* projecting cylinder** of a given curve is a right cylinder whose elements are the *H* or *V* projecting lines of the given curve. The base, in *H* or *V*, of the projecting cylinder is the corresponding projection of the given curve.

Any cylinder with elements perpendicular to *H* or *V* is commonly referred to as a **projecting cylinder.** Any and all curves lying in the surface of a projecting cylinder will have their corresponding projections coinciding in the curve of the base of the cylinder in *H* or *V*.

**104.** Problem 24. **To assume a point on the surface of a given cylinder.**

**Analysis:**

A cylinder will be described orthogonally by at least one section or base, the axis, and the limiting or contour elements in each projection.

A line through any point of the known section or base and parallel to the axis is an element of the cylinder.

Any point on any position of the generating line is a point on the surface of the cylinder.

**Solution** (*a*):

An element of the cylinder may be assumed by passing through any point on the given base or section a line parallel to the given elements, or axis. Any point on this element is a point on the surface of the cylinder.

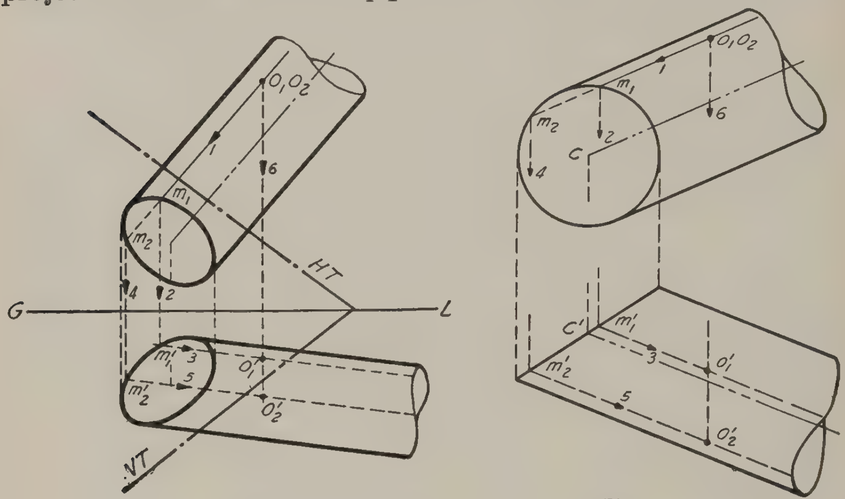
**Solution** (*b*):

If one projection of a point is given, the corresponding projection of the element through this point may be drawn parallel to the axis. Since this element must intersect the curve of the base at a point in the plane of the base, the element and the point on the element are completely determined.

**Construction:**

Given: The cylinders of Figs. 6(*a*) and 6(*b*).

Required: The *V* projections of points $O_1$ and $O_2$ whose *H* projections are assumed at $o_1o_2$.



(*a*) (*b*)

Fig. 6.

If point $O_1$ is a point on the surface of the cylinder, $o_1m_1$ (*1*), which is parallel to the contour elements, must be the *H* projection of the element through $O_1$, and $M_1$ is the point in which it intersects the base. The *V* projection of element $O_1M_1$ passes through $m_1'$ and parallel to the *V* projections of the contour elements (*2*,*3*). The *V* projection of $O_1$ is then at $o_1'$ (*6*).

## CONES

**105.** A *cone* is *generated* by a *straight line* moving so that it *constantly touches a given curve* and *passes through a fixed point* not in the plane of the curve, Fig. 7.

The given curve is called the **directrix,** the given point the **apex,** and the different positions of the generating line the **elements** of the cone.

If a *given plane intersects all the elements of the cone,* the *curve of intersection* is called the **base** of the cone *in that plane.*

The *straight line joining the apex with the center of the base* is called the **axis** of the cone.

A cone is called a **right cone** or an **oblique cone,** depending on whether the *axis* is *perpendicular* to or *oblique* to the *plane of the base.*

The shape of the base and relative position of the axis of a cone are usually indicated by designating it as **right-circular, oblique-elliptical,** etc.

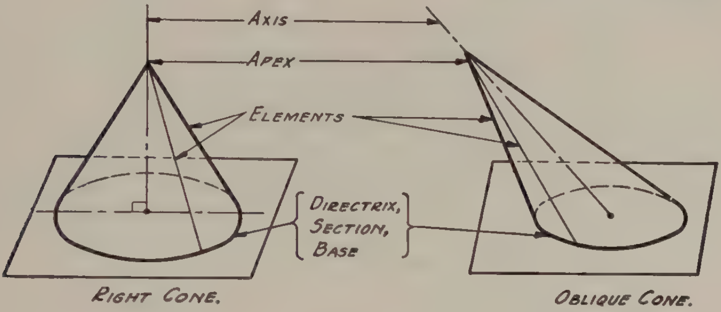


Fig. 7.

106. Problem 25. **To assume a point on the surface of a given cone.**

**Analysis:**

A cone is described orthogonally by at least one section or base, the axis, and the two contour elements in each projection.

All positions of the generating line lie wholly on the surface of the cone.

A line through any point of the known section or base and the intersection of two known elements, or apex, is an element.

Any point on any position of the generating line lies on the surface of the cone.

**Solution:**

An element of the cone may be assumed by connecting any point of a known section or base with the apex. Any point of this element lies on the surface of the cone.

**Construction:**

Given: Cone with apex $O$ and horizontal section, Fig. 8.

Required: The $H$ projections of two points whose $V$ projections are assumed as $a_1'a_2'$.

If point $A_1$ is a point on the surface of the cone, $o'a'_1$ *(1)* is the $V$ projection of the element through that point. The $H$ projection of this element is $om_1$ *(2,3)*, and the $H$ projection of $A_1$ is $a_1$ *(4)*. The $H$ projection of element $OA_2$ is $om_2$ *(2,3)*, and the $H$ projection of $A_2$ is $a_2$ *(4)*.

## THE θ CONE; THE ϕ CONE; THE π CONE

**107.** A **θ cone** is a *right circular cone* whose *axis is perpendicular to H* and whose *base angle is equal to the angle θ of a line or plane,* either given or required, Fig. 9. A **ϕ cone** is a *right circular cone* whose *axis is perpendicular to V* and whose *base angle is equal to the angle ϕ of a line or plane.*

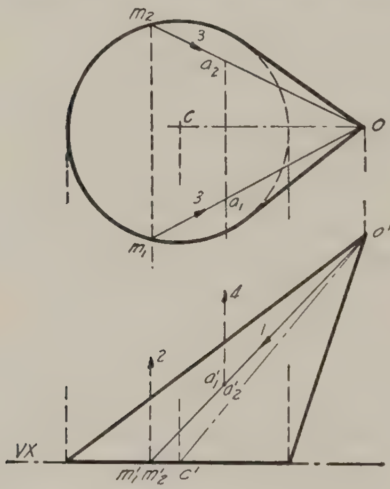


Fig. 8.

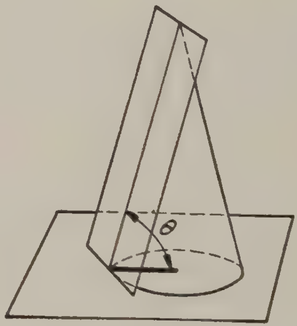


Fig. 9.

A **π cone** is a *right circular cone* whose *axis is perpendicular to P* and whose *base angle is equal to the angle π of a line or plane.*

**108.** Problem 26. **To determine the second projection of a line when given one projection and the angle θ, ϕ, or π.**

**Analysis:**

The line is an element of a θ, ϕ, or π cone whose base angle is equal to the given angle.

The apex of the cone may be at any point of the line.

The second point required to establish the line completely is the intersection of the element of the cone with the base.

**Solution:**

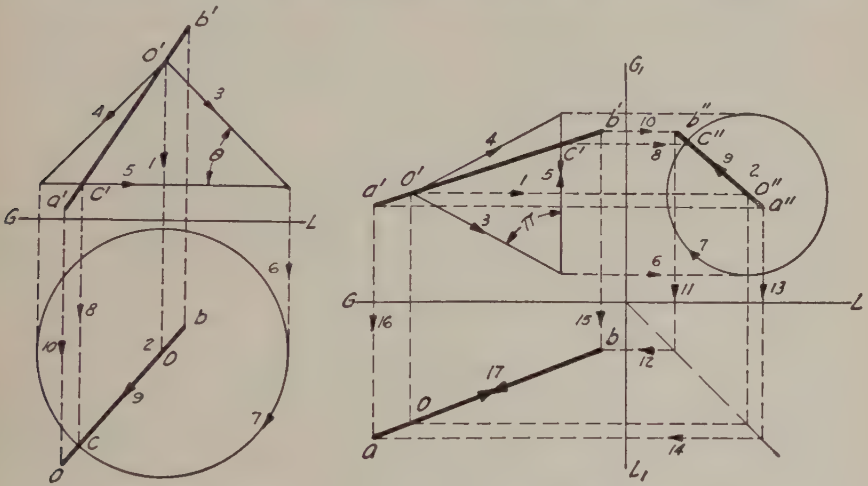
If any point of the line is made the apex of a $\theta$ cone, the line becomes an element of that cone, and its intersection with any base is the second point required to define the line and determine the direction of the required projection.

**Construction:**

Given: $V$ projection of line $AB$ and $\theta = 45$ degrees, Fig. 10(*a*).

Required: $H$ projection of $AB$.

Any point $O$ on $AB$ is chosen as the apex of the $\theta$ cone; the $H$ projection of $O$ may be at any distance above or below $GL$ on the perpendicular from $o'$ to $GL$ (*1*,*2*). The $V$ projection of the $\theta$ cone is the isosceles triangle whose vertex is $o'$, base angles 45 degrees (*3*,*5*), and the $H$ projection is the circle whose center is $o$ and whose diameter is equal to the base of the triangle (*6*,*7*).



(*a*) (*b*)

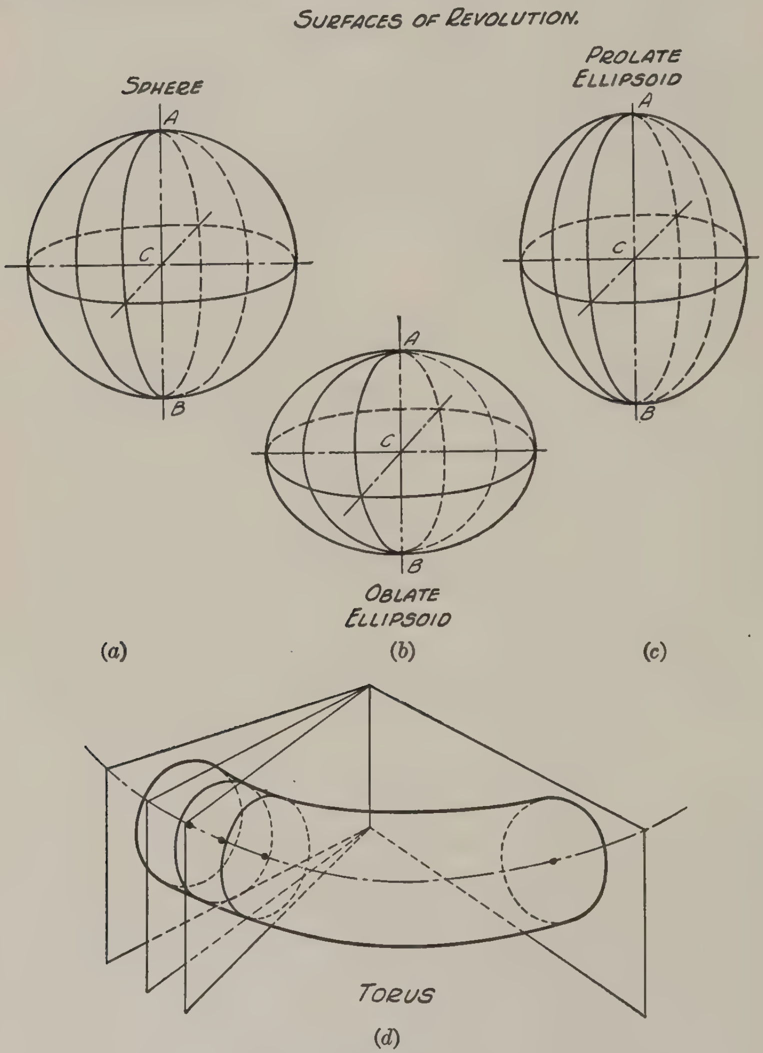
FIG. 10.

Since the line is an element of this cone, the $V$ projection of the point in which the element $AB$ intersects the base of the cone is at $c'$, where $a'b'$ intersects the base of the triangle; its $H$ projection is at $c$, and the $H$ projection of $AB$ is determined (*8*,*10*).

If the angle $\phi$ or $\pi$ is given, the $\phi$ or $\pi$ cone must be used instead of the $\theta$ cone. Figure 10(*b*) illustrates the construction when $\pi$ is given instead of $\theta$.

## DOUBLE CURVED SURFACES

**109. Double curved surfaces** are generated by curves of *single curvature* moving according to some law.



(*a*) (*b*) (*c*)

(*d*)

FIG. 11.

A double curved **surface of revolution** may be generated by revolving a given curve about a given straight line as an axis, Figs. 11(*a–d*). The plane of the given curve may or may not contain the axis of revolution. If it contains the axis, then a plane through the axis intersects the surface in a curve identical with the generating curve.

If a *circle is revolved about one of its diameters as an axis*, the *surface generated* will be a **sphere,** Fig. 11(*a*).

If a *circle is revolved about a line in the plane of the circle but not intersecting the circle*, the *surface generated* is called a **torus,** Fig. 11(*d*).

If an *ellipse is revolved about its major axis*, the *surface generated* will be a **prolate spheroid.** If *revolved about the minor axis*, the *surface* will be an **oblate spheroid,** Figs. 11(*b*) and (*c*).

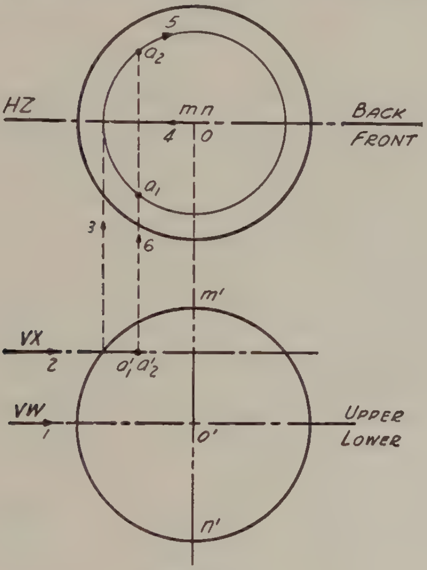


FIG. 12.

**110.** Problem 27. **To assume a point on a surface of revolution.**

**Analysis:**

A surface of revolution is produced by the rotation of one line, straight or curved, about a straight line as an axis.

Each point of the generating line describes a circle whose center is on, and whose plane is perpendicular to, the axis.

Any plane perpendicular to the axis will intersect the surface, if at all, in a circle.

Any point of the above circle is a point on the surface of revolution.

**Solution:**

If one projection of the point is assumed within the limits of either projection of the surface of revolution and a plane is passed through this point perpendicular to the axis of revolution, the projections of the point will lie on the respective projections of the circle cut from the surface by this plane.

**Construction:**

Given: The projections of a sphere with center at $O$, Fig. 12.

Required: The projections of two points on the upper portion of the surface.

All the sphere above the horizontal plane $W$ (*1*) constitutes the **upper** portion of the surface.

Point $a_1'a_2'$ is chosen above $VW$ as the $V$ projections of two points $A_1$ and $A_2$. The horizontal plane $X$ is passed through these points, cutting from the surface a horizontal circle (*2,5*). The $H$ projection of points $A_1$ and $A_2$ are then at $a_1$ and $a_2$ respectively (*6*).

## Chapter VIII

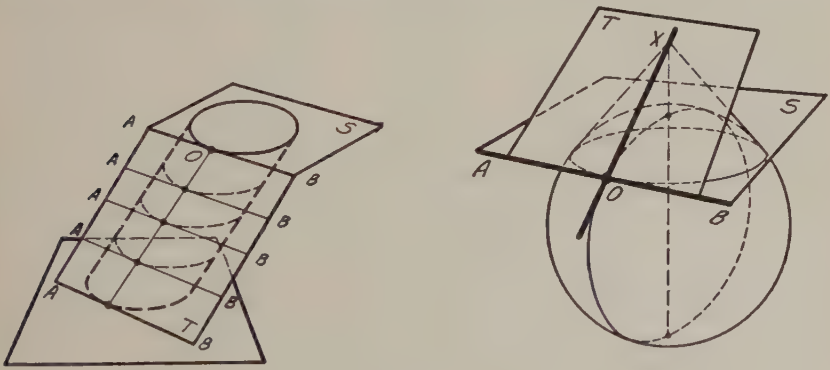
# TANGENT PLANES

### DEFINITIONS

**111.** *A line, AB,* Figs. 1(*a*) and 1(*b*), *is tangent to a surface at a given point if it is tangent to any line on the surface at that point.*

*The projections of a line which is tangent to a curve are tangent to the respective projections of the curve* [Article 100].

*A plane, T,* Figs. 1(*a*) and (*b*), *tangent to a surface at a given point is the locus of all right lines tangent to the surface at that point.*



(*a*)

(*b*)

Fig. 1.

*If a plane, T, is tangent to a surface at a given point, then any other plane, S, passed through this point cuts a straight line, AB, from the tangent plane and a line from the surface such that these two lines are tangent at the given point, O, Figs. 1(a) and (b).*

### PLANES TANGENT TO SINGLE CURVED SURFACES

**112.** Problem 28. **To pass a plane through a given point on a given cone or cylinder and tangent to the surface.**

**Analysis:**

A plane is defined by any two lines in it.

A plane which is tangent to the surface of a cone or cylinder contains one of the elements of the surface. This line is one of the two required to define the plane.

A line tangent to the surface at any point of the line of tangency is the second line required to define the plane.

A line is tangent to a surface if it is tangent to any line on the surface [Article 111]. The second required line above must therefore pass through some point of the line of tangency and be tangent to some convenient curve of the surface through that same point.

**Solution:**

The element of the surface through the given point, Fig. 2(*a*), provides one line for the definition of the plane. The line tangent to the curve of the base through the point of intersection of the element with the base is a second line. The required plane is defined by the tangent and the element.

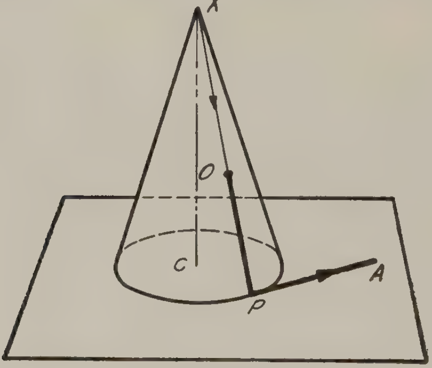


Fig. 2(*a*).

**Construction:**

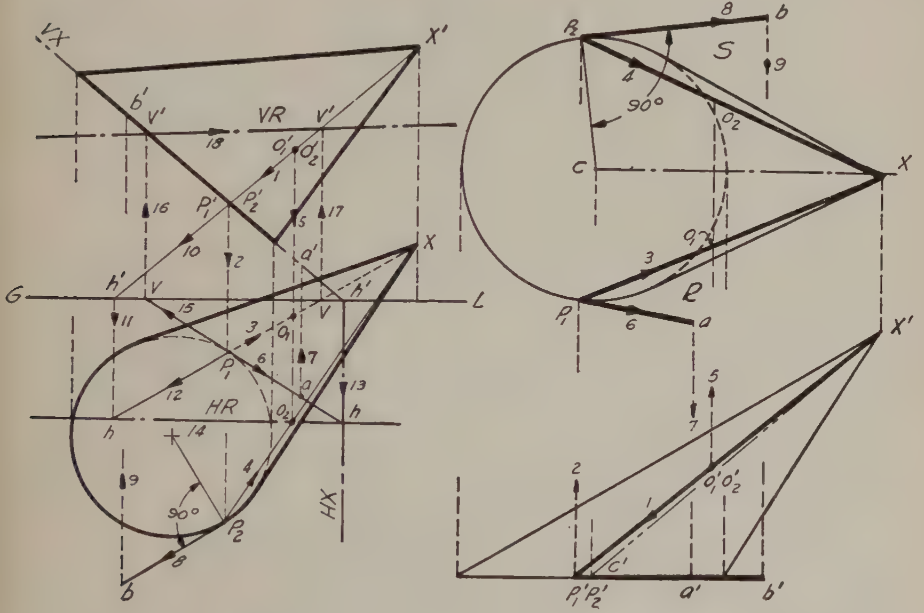
Given: Cone $X$ and the $V$ projections $o_1'$ and $o_2'$ of two points on the cone, Figs. 2(*b*) and 2(*c*). (Base in $V$ projecting plane.)

Required: Planes through points $O_1$ and $O_2$ tangent to cone.

The $V$ projections of the elements $XO_1$ and $XO_2$ are drawn through $o_1'o_2'$ and $x'$ and extended until they intersect the $V$ projection of the base at $p_1'p_2'$ (*1*); $p_1'$ and $p_2'$ are the $V$ projections of the two points in which the elements pierce the plane of the base. Their $H$ projections are at $p_1$ and $p_2$ and the $H$ projections of the elements are $p_1x$ and $p_2x$ (*2,4*).

The lines tangent to the base at $P_1$ and $P_2$ are $P_1A$ and $P_2B$ (*6,9*). The required planes $R$ and $S$ are then defined by lines $XP_1$, $P_1A$ and $XP_2$, $P_2B$, respectively.

The traces of plane $R$, Fig. 2(*b*), are $HR$ (*10,14*) and $VR$ (*15,18*).



(*b*) (*c*)

FIG. 2.

113. Problem 29. **To pass a plane tangent to the surface of a given cone or cylinder through a given point in space.**

**Analysis:**

A plane tangent to the surface of a cone must pass through the apex. The line through the apex and the given point is therefore one line of the required plane.

If the above line is extended to pierce the plane of any section or base of the cone, the line from this piercing point and tangent to the section is tangent to the cone [Article 111] and is a second line of the required plane.

**Solution:**

A straight line through the given point and the apex of the cone, Fig. 3(*a*), is one line of the required plane. If this line is extended until it pierces the plane of a section of the cone and through this piercing point a line is passed tangent to the section, this tangent line and the line through the given point determine the required plane. The element through the point of tangency of the second line to the curve of the base is the element of tangency and must also lie in the tangent plane.

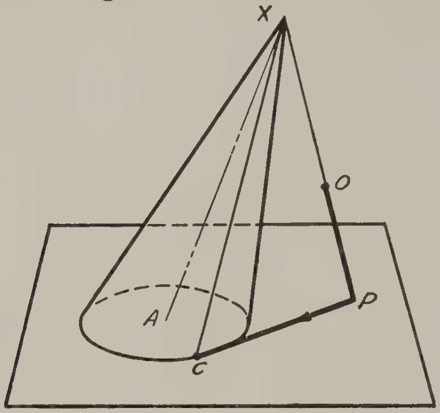


FIG. 3(*a*).

**Construction** (*a*):

Given: Cone $X$ and point $O$, Figs. 3(*b*) and 3(*c*).

Required: Plane through point $O$ tangent to cone $X$.

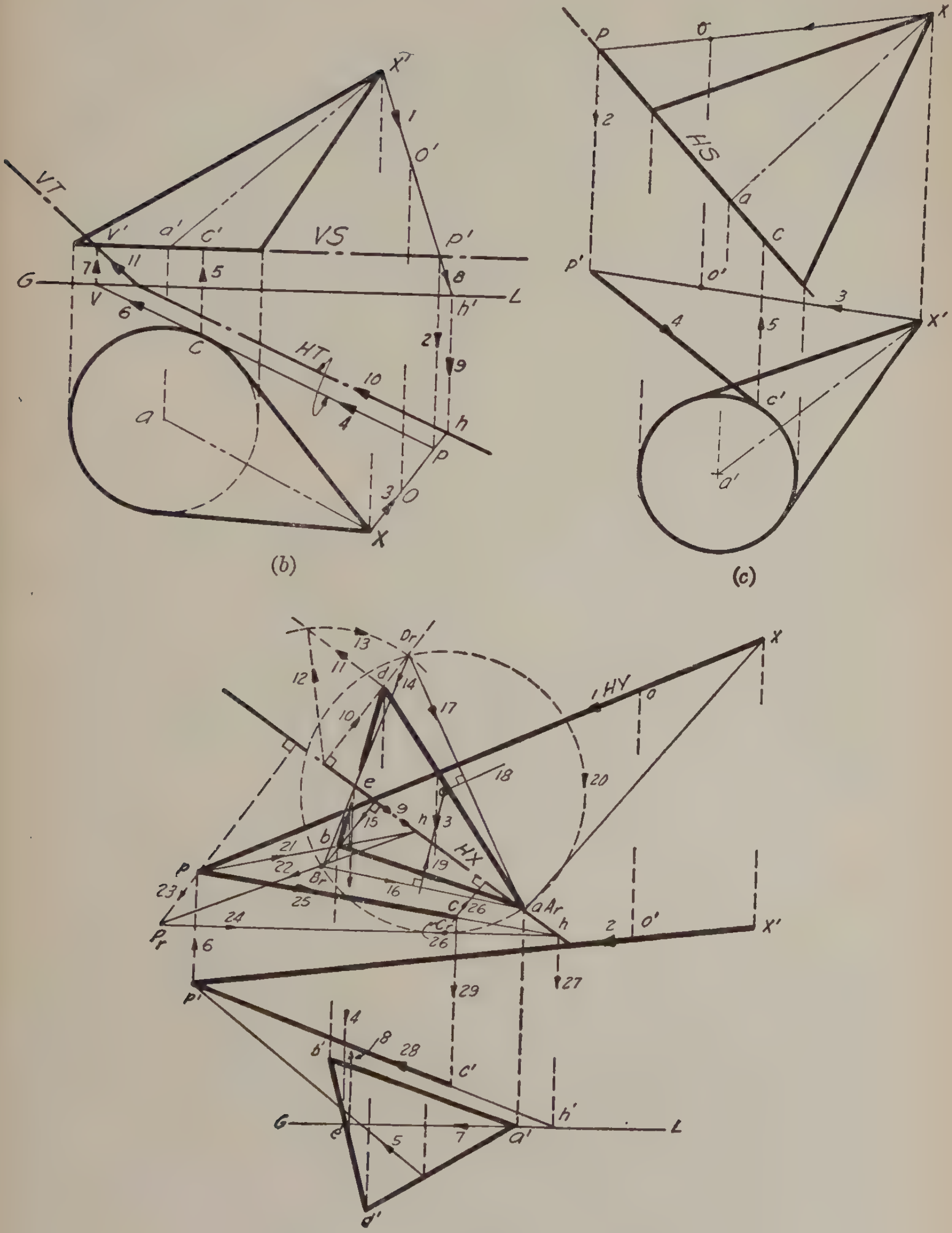
A straight line is passed through $O$ and $X$. It pierces the plane of the base at $P$ (*1,3*). The line $PC$ is passed through $P$ tangent to the base of the cone (*4,5*). The required plane $T$ is defined by the lines $PX$ and $PC$.

The traces of the tangent plane, Fig. 3(*b*), are $HT$ and $VT$ (*6,11*).

**Note:** If the given surface is a cylinder, the above analysis applies equally to it since a cylinder may be considered a cone with its apex infinitely distant from the base. The line through the given point is then parallel to the elements of the cylinder.

**Construction** (*b*):

Given: Point $O$, and point $X$ which is the apex of a cone of circular base lying in the plane of and passing through the three points $A$, $B$, and $D$, Fig. 3(*d*).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(b)

(c)

(d)

FIG. 3.

Required: A plane tangent to cone $X$ and passing through point $O$.

A line is drawn through points $X$ and $O$ (*1,2*). Its piercing point in the plane of $ABD$ is $P$ (*3,6*). A horizontal line, $AE$, is chosen in $ABD$ (*7,9*), and $ABD$ is revolved about $AE$ into a horizontal position $A_rB_rD_r$ (*10,17*). The circle is then drawn through $A_r$, $B_r$, and $D_r$ (*18,20*). A line is drawn from $P$ through $B$ to intersect $HX$ at $H$ (*21*). The revolved position of this line is $hB_rP_r$ (*22,23*). The revolved position of the tangent $PC$ is then drawn from $P_r$ tangent to the circle at $C_r$ (*24*). The $H$ projection of $PC$ is $pc$ (*25,26*) and the $V$ projection, $p'c'$ (*27,29*). The required plane is $XPC$.

**114.** Problem 30. **To pass a plane tangent to a given cone and parallel to a given line.**

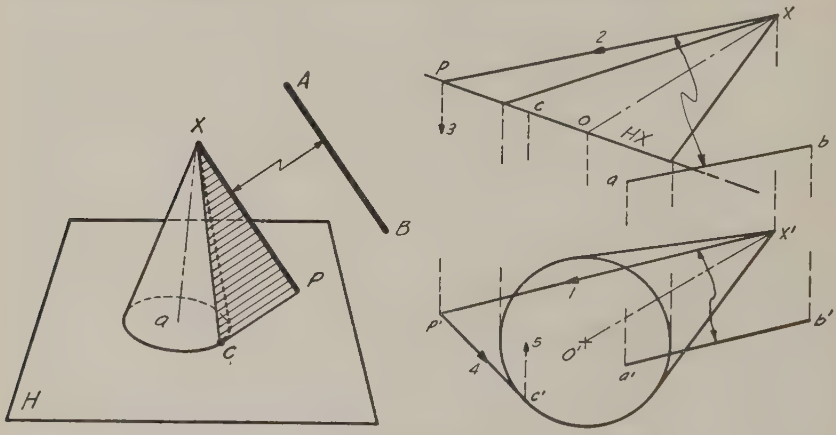
**Analysis:**

The required plane must pass through the apex of the cone in order to be tangent to the cone.

The plane must contain a line parallel to the given line in order to be parallel to the line.

One of the two lines required to define the plane must therefore be a line through the apex and parallel to the given line.

The second line must pass through some point of this auxiliary line and be tangent to the surface of the cone [Article 111].



(*a*) FIG. 4. (*b*)

**Solution:**

A straight line through the apex of the cone parallel to the given line, Fig. 4(*a*), is one line of the required plane. If this line is extended until it pierces the plane of a section or base of the cone, a straight line through this piercing point tangent to the section is tangent to the cone and is a second line of the required plane. The required plane is then defined by this tangent line and the line passed through the apex. The element through the point of tangency of the second line to the curve of the base is the element of tangency and must also lie in the tangent plane.

**Construction** (*a*):

Given: Cone $X$ and line $AB$, Figs. 4(*b*) and 4(*c*).

Required: Plane tangent to cone and parallel to $AB$.

The line $XP$ through the apex $X$ parallel to the given line $AB$

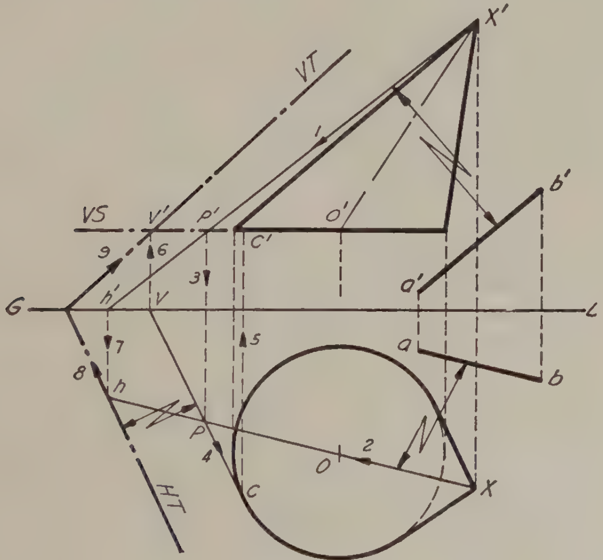


Fig. 4(*c*).

pierces the plane of the base of the cone at $P$ (1,3). The line $PC$ is passed through $P$ tangent to the base of the cone at $C$ (4,5). The required plane $T$ is defined by the two lines $PX$ and $PC$.

The traces of plane $T$, Fig. 4(*b*), are $HT$ and $VT$ (6,9).

**Construction** (*b*):

Given: Cone with apex $X$ as of Fig. 3(*d*), and any line parallel to $XO$.

Required: A plane tangent to the cone and parallel to the line.

The construction (*b*), Fig. 3(*d*), for the cone and a point, $O$, applies entirely to the above problem. That is, a line passed through $X$ parallel to the given line, Fig. 3($d$), is $XO$. The plane $XPC$ is then the required plane.

**115.** Problem 30: Example 1. **To pass a plane tangent to a given cylinder and parallel to a given line.**

**Analysis:**

The given line and a line through any point of it and parallel to the elements of the cylinder define a plane parallel to the required plane.

The above plane and the required plane will intersect the plane of any section or base of the cylinder in parallel lines.

A line tangent to the base and parallel to the intersection of the auxiliary plane with the plane of the base is one line of the required plane.

The above tangent line and the element of the cylinder through the point of tangency define the required plane.

**Solution:**

If through any point of the given line, Fig. 5($a$), a line parallel to the elements of the cylinder is passed, the plane of these two lines is parallel to the required plane. A line tangent to the base of the cylinder and parallel to the line of intersection of the plane just defined with the plane of the base is one line of the required plane. The element through the point of tangency is another. The required plane is defined by these two intersecting lines.

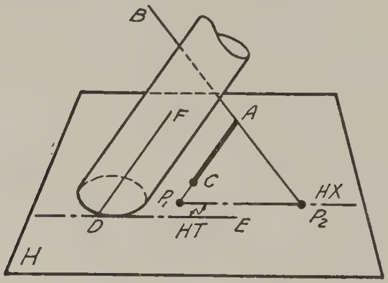


Fig. 5($a$).

**Construction:**

Given: Oblique cylinder and line $AB$, Fig. 5($b$).

Required: A plane tangent to the cylinder and parallel to $AB$.

Line $AC$ is drawn through point $A$ parallel to the elements of the cylinder (*1,2*), defining a plane $CAB$ parallel to the required plane. Lines $AC$ and $AB$ pierce the plane of the base of the cylinder at points $P_1$ and $P_2$ (*3,6*), giving the line $P_1P_2$ as the intersection of plane $CAB$ with plane $T$ of the section of the cylinder. A tangent $DE$ (*7,9*) is drawn to the section and parallel to $P_1P_2$. Element $DF$ (*10,12*) then completes the required tangent plane $FDE$ parallel to $AB$.

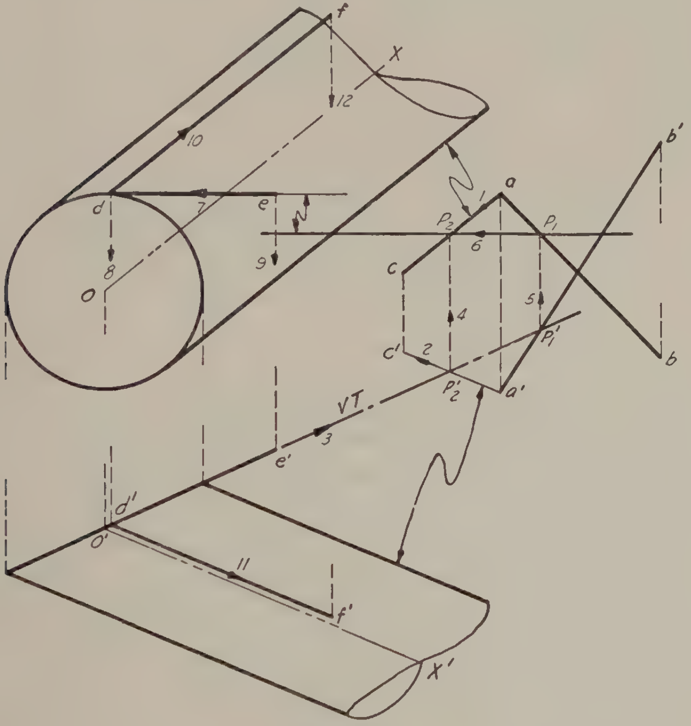


FIG. 5(*b*).

**116.** Problem 30: Example 2. **To pass a plane through a given line and making a specified angle with a given plane.**

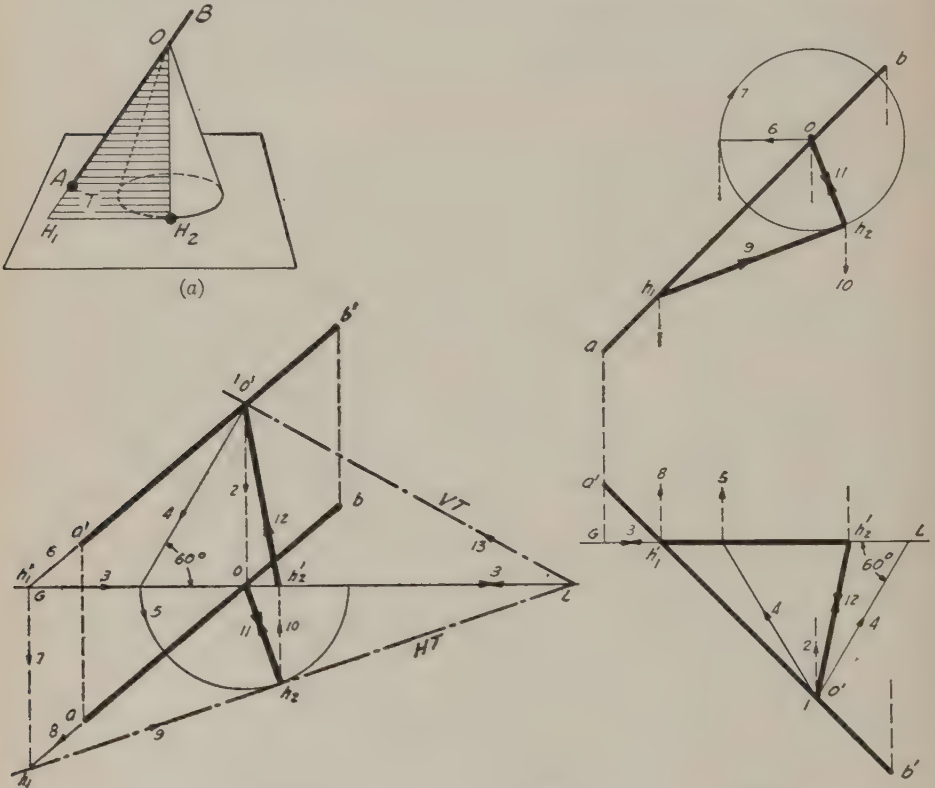
**Analysis:**

One additional line intersecting or parallel to the given line is required to define the required plane.

The required plane will be tangent to a $\theta$, $\phi$, $\pi$, or $X$ cone whose apex is on the given line. The element of tangency is the additional line required to define the plane. (**Note:** The angle $\theta$, $\phi$, $\pi$, or $X$ of the plane cannot be less than that of the line.)

**Solution** (Given $\theta$):

$O$, any point of the given line, Fig. 6($a$), is made the apex of a right circular cone whose axis is vertical and base angle equal to the given angle $\theta$. The line drawn from the point in which the given line pierces the plane of the base of the cone tangent to the base, will determine a point on the element of tangency. The given line and the tangent to the base, or the given line and the element of the cone, will determine the required plane.



($b$) ($c$)

FIG. 6.

**Construction:**

Given: Line $AB$ and $\theta$ equal to 60 degrees, Figs. 6($b$) and 6($c$).

Required: Plane $T$ through $AB$ with $\theta$ equal to 60 degrees.

Any point $O$ is selected on $AB$ as the apex of the $\theta$ cone (*1,2*). The $\theta$ cone is constructed with its elements of any convenient length (*4,5*), Fig. 6(*b*)—(*4,7*), Fig. 6(*c*). The plane of the base of the cone may be regarded as $H$, hence the $V$ projection of the base lies in $GL$. $AB$ pierces the plane of the base of the cone in point $H_1$,(*6,8*), Fig. 6(*b*)—(*8*), Fig. 6(*c*). The tangent from $H$ to the base of the cone is $H_1H_2$ (*9,10*) and the element of tangency is $OH_2$ (*11,12*). Points $OH_1H_2$ and the lines which connect them then define the required plane $T$. The traces of plane $T$ in Fig. 6(*b*) are $HT$, $VT$ (*9* and *13*).

## PLANES TANGENT TO DOUBLE CURVED SURFACES

117. Problem 31. **Through a given point on a given double curved surface of revolution to pass a plane tangent to the surface.**

### Analysis:

The plane will be defined by two lines tangent to the surface and intersecting at the given point of tangency [Article 111].

A plane through the given point and perpendicular to the axis of revolution will intersect the surface in a circle. A tangent to this circle is tangent to the surface [Article 111].

A right circular cone can always be constructed with its apex on the axis of revolution and tangent to the surface of revolution on any circle cut from it by a plane perpendicular to the axis. All elements of such a cone are tangent to the surface at points on the circle.

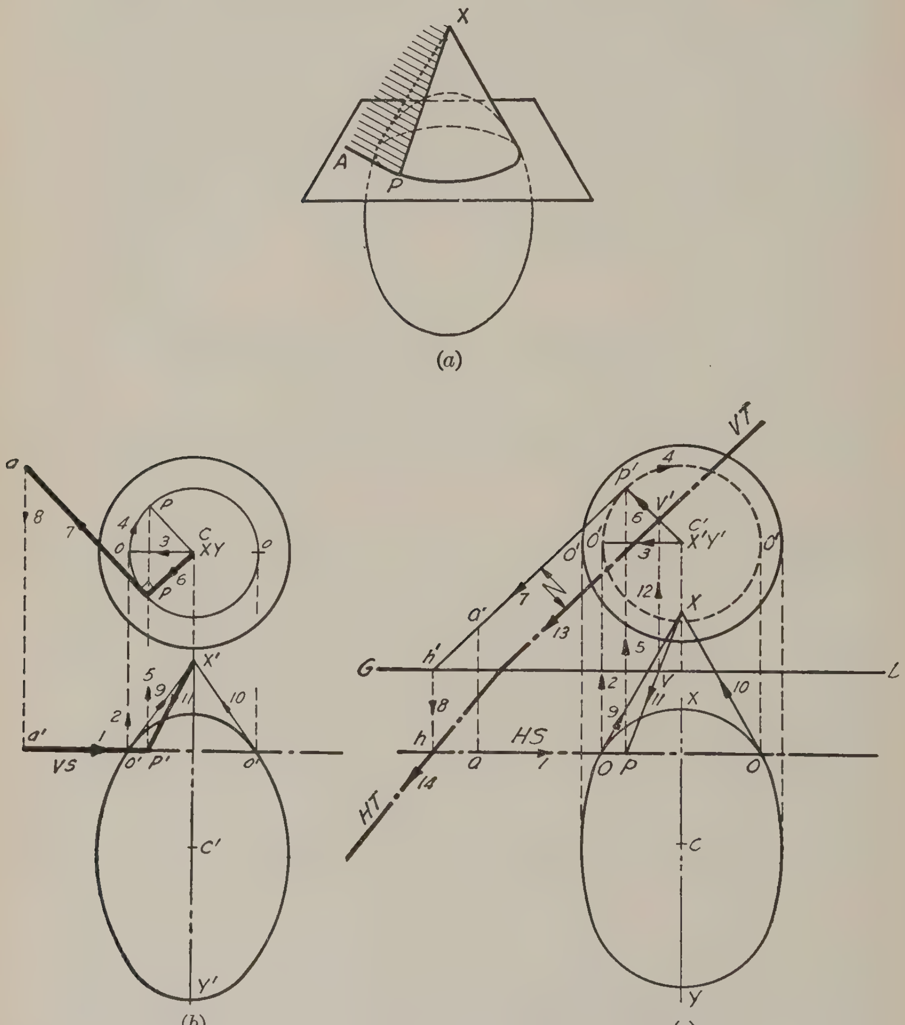
### Solution:

If a plane is passed through the given point, Fig. 7(*a*), perpendicular to the axis of revolution, it will intersect the surface in a circle. The tangent to this circle at the given point is one of the required lines. If then a right circular cone is constructed with its apex on the axis of revolution and tangent to the surface on the above circle, the element through the given point is the second line required to define the tangent plane.

**Construction:**

Given: Spheroid and point $P$ on spheroid, Figs. 7($b$).

Required: Plane through $P$ tangent to spheroid.



($a$)

($b$)

($c$)

Fig. 7.

Plane $S$ through point $P$ perpendicular to the axis of revolution intersects the surface in a circle whose $V$ projection is the chord $o'o'$ and whose $H$ projection is the circle with its center at *c* and diameter equal to *o′o′* (*1,4*). The line *PA* is passed through *P* tangent to this circle (*5,8*). A right circular cone is constructed with its apex *X* on the axis of revolution, and tangent to the spheroid along the circle cut from the surface by plane *S* (*9,10*). Its *V* projection, Fig. 6(*b*), is the isosceles triangle *x′o′o′*, the sides *x′o′* and *x′o′* being tangent to the ellipse at *o′* and *o′*; the *H* projection is the circle *opo*. Since all the elements of this cone are tangent to the spheroid, the element *XP* (*6,11*) is the second of the two required tangent lines. The required plane *T* is defined by the tangent lines *PA* and *PX*, Fig. 7(*b*).

The traces of plane *T*, Fig. 7(*c*), are *HT* and *VT* (*12,14*).

### **118.** Problem 31: Example 1. **Through a given point on a given sphere to pass a plane tangent to the sphere.**

**Analysis:**

The analysis presented for passing a plane tangent to a surface of revolution [Article 117] applies for the sphere.

An additional fact of importance in the case of the sphere is that the radius of the sphere passing through its point of tangency with a plane is perpendicular to the plane.

**Solution:**

If a plane is passed through the given point, Fig. 8(*a*), perpendicular to the radius of the sphere to that point, it will be tangent to the sphere.

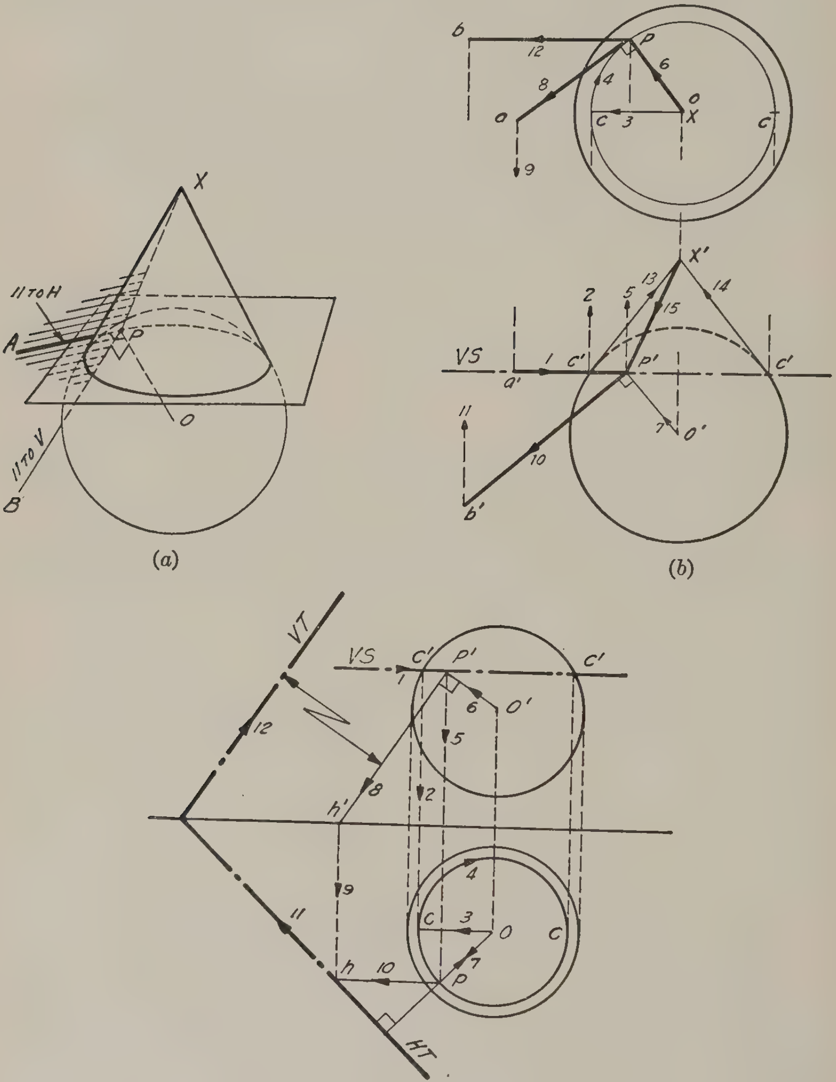
**Construction:**

Given: Sphere *O* and point *P* on the surface, Fig. 8(*b*).

Required: Plane through *P* tangent to the sphere.

The *V* projection *p′* of point *P* is given. The *H* projection is determined by passing through *P* a horizontal plane *S* (*1*), which intersects the sphere in a circle whose *V* projection is *c′c′* and *H* projection a circle with center at *o* and diameter equal to *c′c′* (*2,4*). Since point *P* lies on this circle, its *H* projection is at *p* (*5*). *OP* (*6,7*) is the desired radius of the sphere. The required plane *T* passes through point *P* perpendicular to radius *OP* and is defined by the *horizontal AP* (*8,9*) and the *vertical BP* (*10,12*). The plane is defined also by $AP$ and $PX$ (*13,15*) as in the case of the ellipsoid of revolution, Fig. 7(*b*).

The traces of plane $T$, Fig. 8(*c*), are $HT$ and $VT$ (*11,12*).



(*a*) (*b*)

(*c*)

Fig. 8.

## CHAPTER IX

# INTERSECTION OF SURFACES

**119.** The line of intersection of any two surfaces is the locus of all points common to the surfaces.

If a line on one surface intersects a line on another surface, their intersection is a point common to the two surfaces.

If two intersecting surfaces are cut by a plane, the lines cut from them may intersect, since they lie in a common plane. If these lines, one on each surface, do intersect, their points of intersection are common to the two given surfaces.

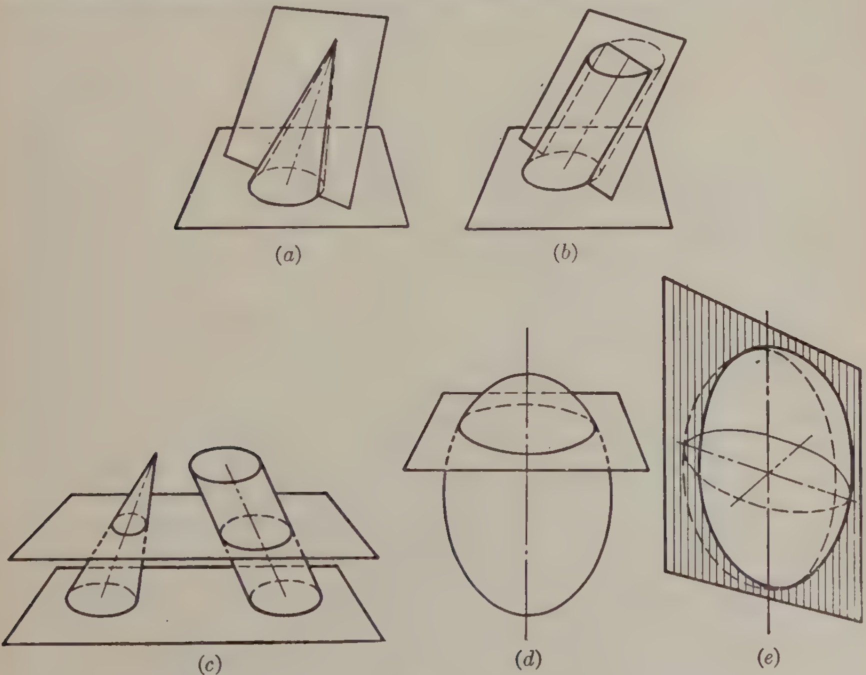

(a) (b) (c) (d) (e)

FIG. 1.

A simple and generally used method of determining the intersection of surfaces involves the passing of planes in such a manner as to intersect the surfaces in lines which project on the coordinate planes either as straight lines or circles. In applying this method the following facts of solid geometry should be remembered, Figs. 1(*a-e*).

*a*. Secant planes passing through the apex of a cone intersect the surface in elements.

*b*. Secant planes parallel to the axis of a cylinder intersect the surface in elements.

*c*. Secant planes parallel to the base of a cone or cylinder intersect the surface in curves similar to or equal to the base.

*d*. Secant planes perpendicular to the axis of a surface of revolution intersect that surface in one or more circles.

*e*. Planes through the axis of a surface of revolution, known as *meridian planes*, intersect the surface in *meridian lines*. All meridian lines of a surface are equal. The surface may be generated by revolution of a meridian line. The projection of the surface on a plane parallel to the axis is equal to a meridian line.

## VISIBILITY

**120.** The various surfaces treated in descriptive geometry are considered opaque. In describing solids on the coordinate planes there is thus a need for conventional methods to designate what is on the side *toward* the observer and hence *visible*, and what on the side *away from* the observer and therefore *invisible*. Points on the surfaces of the various solids may be visible in some views and invisible in others. There arises therefore the need for a method of determining if a point is on the side *toward* or on the side *away from* the observer *for each view*.

When two surfaces are involved in the same problem, points or lines may be on the near side of one surface but concealed wholly or in part by the other. The unseen part of any such line should be so designated by the proper convention, which is a broken line.

In determining which portions of a line of intersection should be represented as visible and which invisible in the $V$ projection, reference to the $H$ projection must generally be made: that is, it is easier to determine from the $H$ projection which part of a line is on the *front* and which on the *back* of a given surface. For example: The ellipsoid, Fig. 2, is intersected by plane $T$ in the line $CFDE$. A part of this line is described as *visible* and the remainder as *invisible* in the $V$ projection. Since the axis of the ellipsoid is vertical, all the surface *before* the median plane $X$ is *visible* in the $V$ projection. That is, plane $X$ intersects the ellipsoid in the curve which is the contour curve on the $V$ plane, hence the *projections of the points where a line on the surface passes from the visible to the invisible must be on the contour of that projection.* Therefore the *$V$ projection of the line must change in visibility where it touches the contour.* In ths case the line is tangent to the $V$ contour at points $e'$ and $f'$. So $e'd'f'$ is *visible* because $D$ is *before* the median plane $X$; and $e'c'f'$ is *invisible* because $C$ is *behind* $X$.

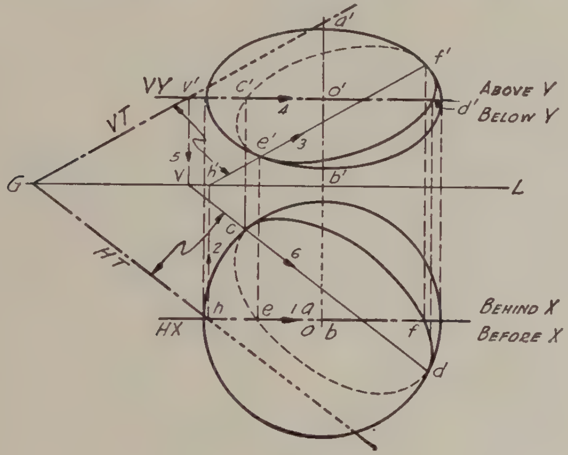


Fig. 2.

In similar fashion, median plane $Y$ cuts the $H$ contour curve from the ellipsoid. *Visibility will change at the tangent points $c$ and $d$,* and the portion of the line $CFDF$ *above* $Y$ is hence *visible* on $H$ while that *below* it is *invisible.* Point $F$ is *above* $Y$, hence $cfd$ is *visible* and $ced$ is *invisible.*

In dealing with ruled surfaces, the determination of visibility of points can best be accomplished by determining the visibility of the *elements* on which the points lie. If a point on the intersection of two surfaces is on elements *both of which* are visible from the *front,* this point is *visible* in the $V$ projection. If, on the other hand, *either* of the two *elements* passing through a certain point is seen to be *invisible*, the *point* is then *invisible* in that projection.

### INTERSECTION OF PLANES AND SINGLE CURVED SURFACES

121. Problem 32. **To find the line of intersection of a given plane with a given cone or cylinder.**

**Analysis:**

The intersection of a plane with a cone or cylinder is defined by a series of points common to the plane and the surface of the cone or cylinder.

Points are derived as the intersections of lines, which in turn define planes.

Points will hence be derived at the desired intervals to define the required line of intersection by passing a series of auxiliary planes such that they will intersect the given plane and cone or cylinder in straight lines. The intersections of each such set of lines will be points of the required line of intersection.

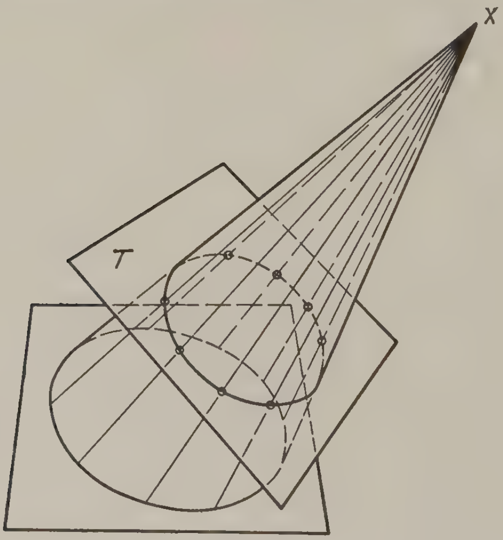


FIG. 3(*a*).

**Solution:**

Elements of the cone or cylinder, Fig. 3(*a*), pierce the given plane $T$ in points common to both surfaces, hence on their line of intersection. The line through these points is the required line of intersection.

**Construction:**

Given: Cone or cylinder and plane $T$, Figs. 3($b,c,d$).

Required: The line of intersection.

The point in which an element of the cone or cylinder pierces the given plane $T$ will be found by passing an $H$ or $V$ projecting plane through the element, finding the line of intersection of the projecting plane and plane $T$, and determining the point in which this line intersects the element [Article 62]. The projecting plane of any element will, in general, cut a second element from the opposite side of the surface whose piercing point in plane $T$ will provide a second point on the required line of intersection. Thus, an economical procedure in construction will be to assume a number of projecting planes at somewhat uniform intervals throughout the range between the outline or contour elements in either view, and to find the piercing points in plane $T$ of both the elements lying in each projecting plane. In this fashion the resulting points will be more or less uniformly distributed around the required intersection. In particular, the piercing points of the four contour elements should be found, as these are the

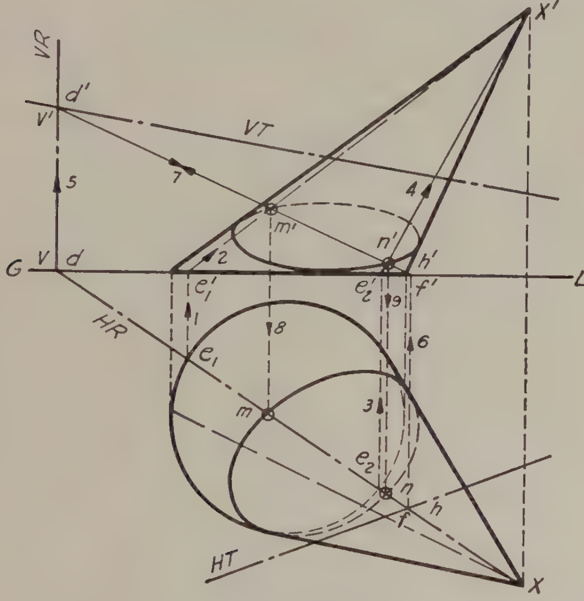


FIG. 3($b$).

points at which the visibility of the line of intersection will change in either view [Article 120].

Projecting plane *R* intersects the cone or cylinder, Figs. 3 (*b*), (*c*), (*d*), in elements through points $E_1$ and $E_2$ (*1,4*), and intersects plane *T* in the line *DF* (*5,7*). Since these two elements and line *DF* lie in plane *R*, they intersect at two points, *M* and *N* (*8,9*), on the required line of intersection.

**Note:** In each of the figures in this chapter, only one auxiliary plane is passed, and the points found by the elements it contains are indicated. The complete construction involves simply a repetition of the same operation with other planes.

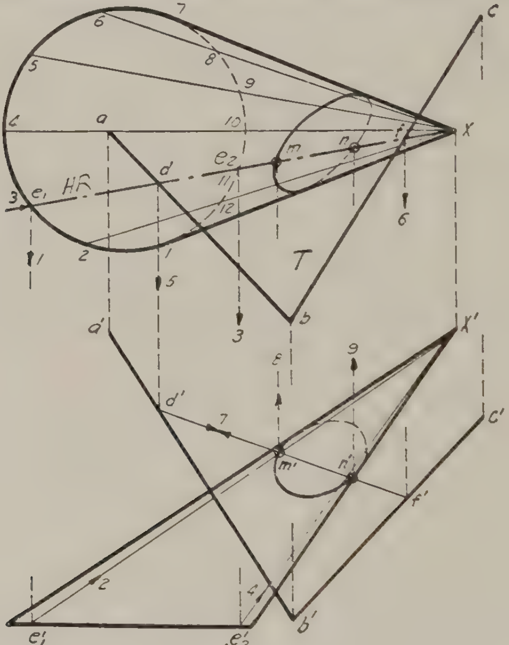


FIG. 3(*c*).

122. Problem 32: Example 1. **To find the points in which a given straight line or plane curve pierces the surface of a given cone or cylinder.**

**Solution:**

If through the given line a plane is passed and its intersection with the given surface determined, the intersections of the given line with this line of intersection will be the required piercing points.

**Construction:**

In finding the piercing points of a straight line in a given cylinder, usually the construction will be simplified by using the $H$ or $V$ projecting plane of the given line, or by passing a plane through the line and parallel to the axis of the cylinder, cutting elements from the surface. Or, if the given surface is a cone, a plane passed through the given straight line and the apex of the cone will cut elements from the surface.

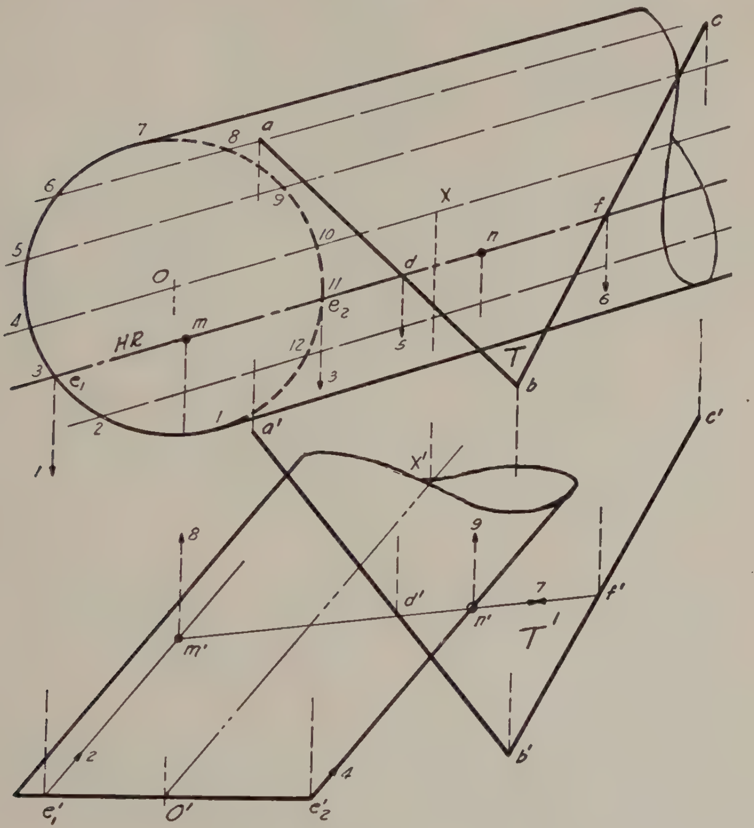


FIG. 3(*d*).

**123.** Problem 32: Example 2. **To determine a line in a given plane making a given angle $X$, $\theta$, $\phi$, or $\pi$, with $T$, $H$, $V$, or $P$.**

**Analysis:**

The maximum angle which a line in a given plane, $S$, may make with another plane, $T$ or $H$, is the angle which the planes make with each other. The minimum angle is zero.

A line of a plane, $S$, which makes with plane $T$ or $H$ a given angle $X$ or $\theta$, may be regarded as an element of an $X$ or $\theta$ cone whose apex is in plane $S$.

The line in question is thus the intersection of the given plane $S$ and an $X$ or $\theta$ cone whose apex is any desired point of plane $S$.

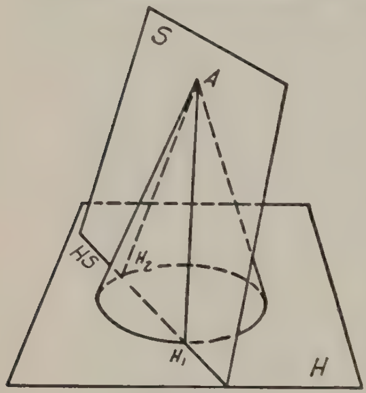
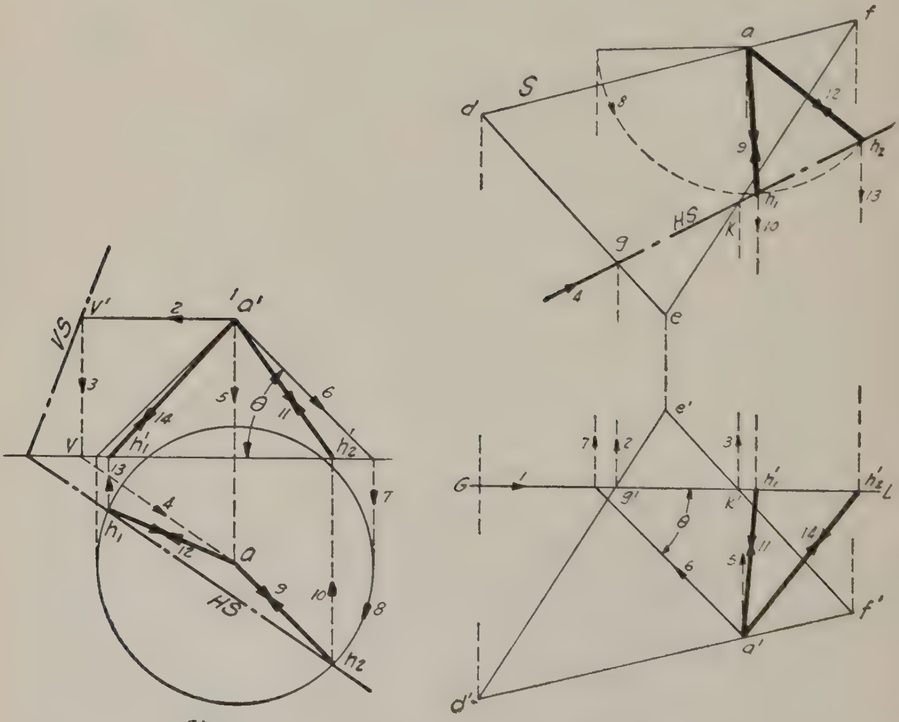


FIG. 4(a).



(b) (c)

FIG. 4.

**Solution:**

If a point $A$ of plane $S$, Fig. 4($a$), is made the apex of a $\theta$ cone, the two elements of intersection of plane $S$ and the cone will make with $H$ the given angle $\theta$.

**Construction:**

Given: Plane $S$, Figs. 4($b$) and 4($c$).

Required: A line in plane $S$, making an angle of 45 degrees with $H$.

Any point $A$ in plane $S$ is made the apex of a $\theta$ cone with base angle of 45 degrees and with base resting in $H$ (*1,8*). Plane $S$ intersects the cone in elements $AH_1$ and $AH_2$ (*9,14*) which are the required lines.

In Fig. 4($c$), $H$ is assumed wherever convenient (*1*) and the $H$ trace of $S$ is determined in this plane (*2,4*).

## INTERSECTION OF PLANES AND DOUBLE CURVED SURFACES

**124.** Problem 33. **To find the line of intersection of a given plane and a given double-curved surface of revolution.**

**Analysis:**

The intersection of the plane and the surface of revolution is defined by a series of points common to the plane and the surface.

Points for this purpose are derived as intersections of coplanar lines.

Points will be derived at the desired interval to define the required line of intersection by passing a series of auxiliary planes such that they will intersect the given surface in circles (i.e., *perpendicular to the axis* of revolution) and the given plane in straight lines. The intersection of each such circle and corresponding straight line provides two points on the required line of intersection.

**Solution:**

Planes perpendicular to the axis of the surface intersect the surface in circles and the given plane in straight lines, Fig. 5($a$). The points of intersection of these straight lines and circles are common to the surface and the given plane, hence are points of the required line of intersection.

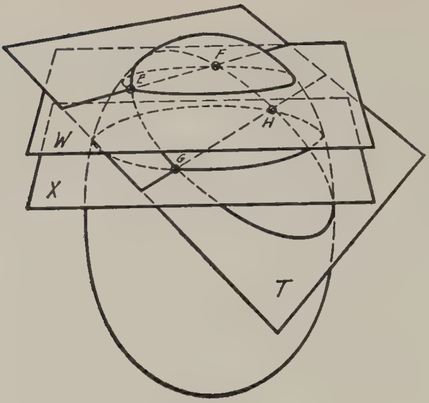

F
E
H
W
G
X
T

Fig. 5(a).

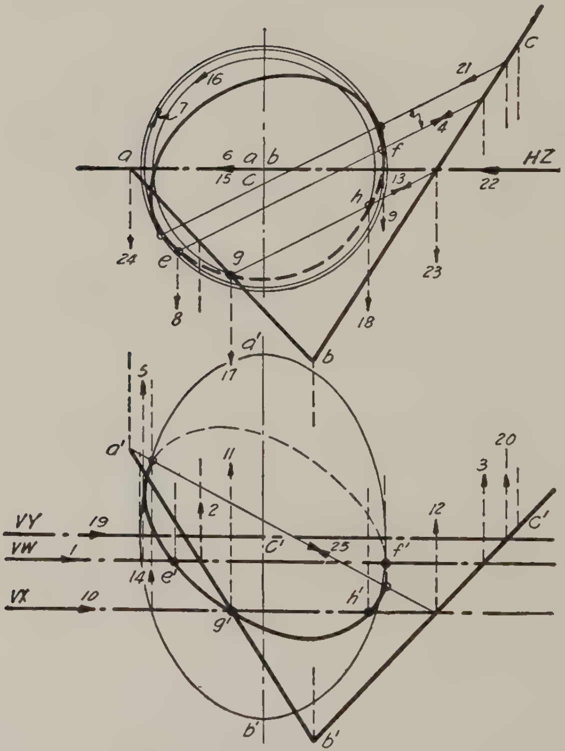

C
21
16
7
4
a
6 a b
f
HZ
15 C
13
22
h
9
24
e
g
23
8
18
a'
b
17
5
20
a'
11
3
2
12
C'
VY
19
VW
1
C'
25
f'
14
e'
VX
10
h'
g'
b'
b'

Fig. 5(b).

**Construction:**

Given: Ellipsoid of revolution and plane $T$ $(ABC)$, Fig. 5(*b*).
Required: The line of intersection.

Plane $W$, perpendicular to the axis, intersects the plane $T$ in a line *(1,4)* and the ellipsoid in a circle *(5,7)*, which in turn intersect in points $E$ and $F$ *(8,9)* of the required line. Plane $X$, also perpendicular to the axis, intersects the plane in a line *(10,13)* and the ellipsoid in a circle *(14,16)*, which in turn intersect in points $G$ and $H$ *(17,18)* of the required line. Additional planes perpendicular to the axis of revolution each provide two points of the required line until the entire line is satisfactorily defined as shown.

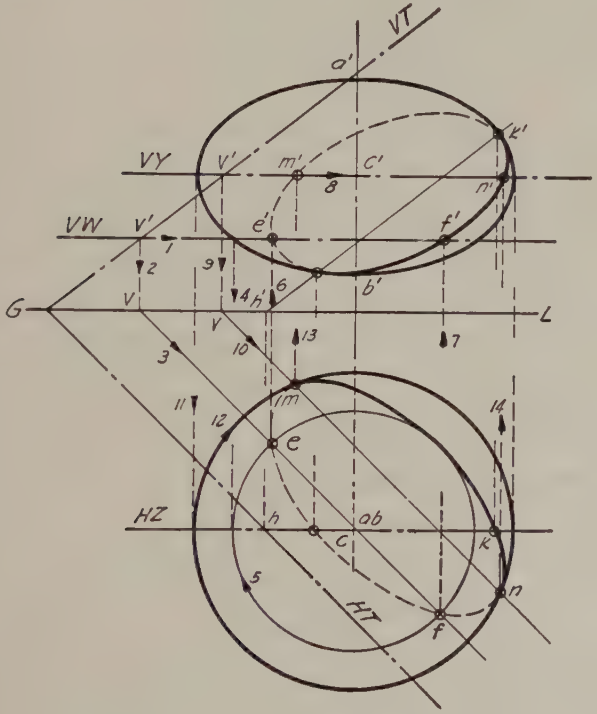


FIG. 5(*c*).

The points at which the respective projections of the line of intersection touch the contour curves are the points of division between the visible and invisible portions of the line. Auxiliary planes should be used deliberately to determine these points; that is, two planes which will intersect the surfaces in the contour curves, as $Y$ ($VY$), and $Z$ ($HZ$), Fig. 5($c$).

**125.** Problem 33: Example 1. **To find the points in which a given straight line or plane curve pierces a surface of revolution.**

**Solution:**

If the intersection of any plane through the line, or the plane of the curve, and the given surface is determined, the intersections of the given line with this line of intersection are the required piercing points.

## INTERSECTIONS OF SINGLE CURVED SURFACES

**126.** Under this classification there may be included the combination of cone with cone, cone with cylinder, and cylinder with cylinder. In any case, the method usually employed is to pass a series of auxiliary planes in such a manner as to intersect the surfaces in elements. This procedure requires, then, that each auxiliary plane contain the apex of the cone and be parallel to the elements of the cylinder, if both are involved; with two cones, each plane must contain the apexes of both; in the case of two cylinders, the planes must be parallel to the elements of both cylinders.

## NUMBER OF CURVES OF INTERSECTION

**127.** Two surfaces may intersect in such ways, Figs. 6, 7, and 8, as to give either one or two closed curves. The shapes of the curves will depend of course upon the nature of the intersecting surfaces and upon their directional relations and their locations with respect to each other. A prime consideration always is the number of curves of intersection and, if there are two, whether they have a common point. These qualities need to be determined as promptly as possible.

The very process of determining points of the curve or curves through the use of common secant planes serves to determine the number and nature of the curves. If, for example, it is determined that two planes, $Y$ and $Z$, passing through the apex of the cone and tangent to the cylinder on opposite sides, intersect the cone in two elements each, the arrangement must be that of

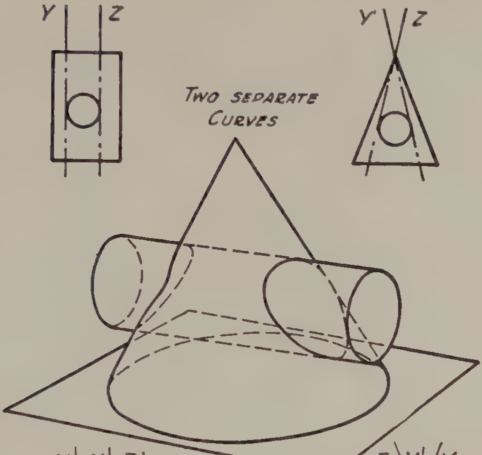


Fig. 6.

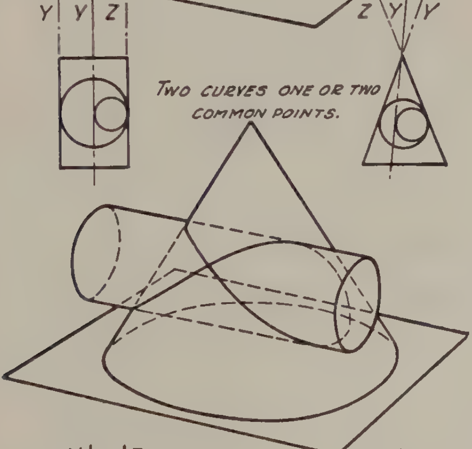


Fig. 7.

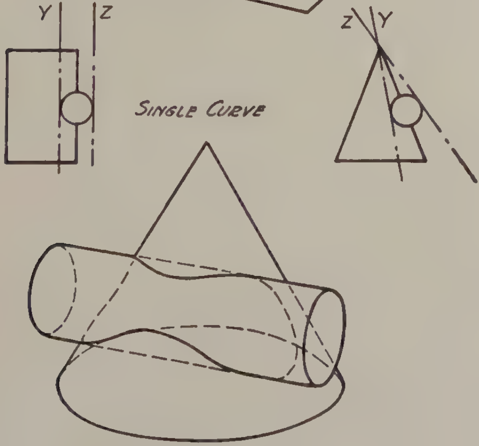
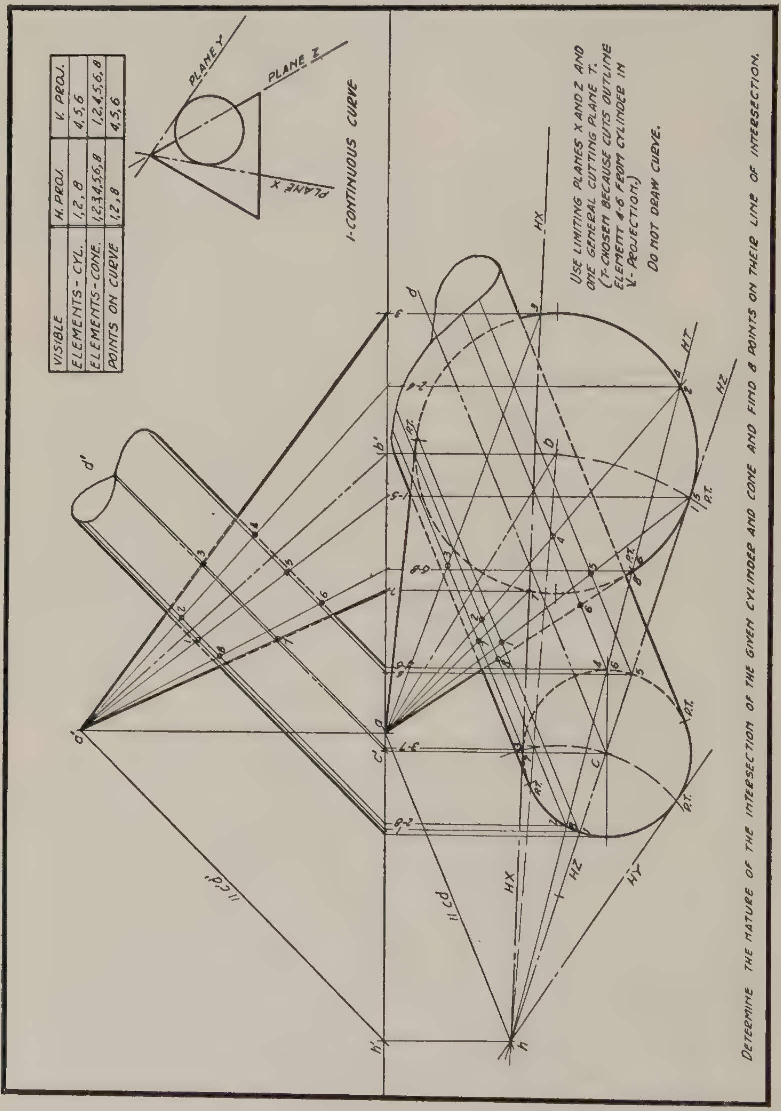


Fig. 8.

Fig. 6, where there are two separate curves. If one of the two planes through the apex and tangent to the two sides of the cylinder is tangent to the cone also, the arrangement is that of Fig. 7, in which there are two curves with one point in common. And finally, if one of the two planes through the apex and tangent to the opposite sides of the cylinder does not intersect the cone



| VISIBLE | H. PROJ. | V. PROJ. |
|---|---|---|
| ELEMENTS - CYL. | 1,2,8 | 4,5,6 |
| ELEMENTS - CONE. | 1,2,3,4,5,6,8 | 1,2,4,5,6,8 |
| POINTS ON CURVE | 1,2,8 | 4,5,6 |

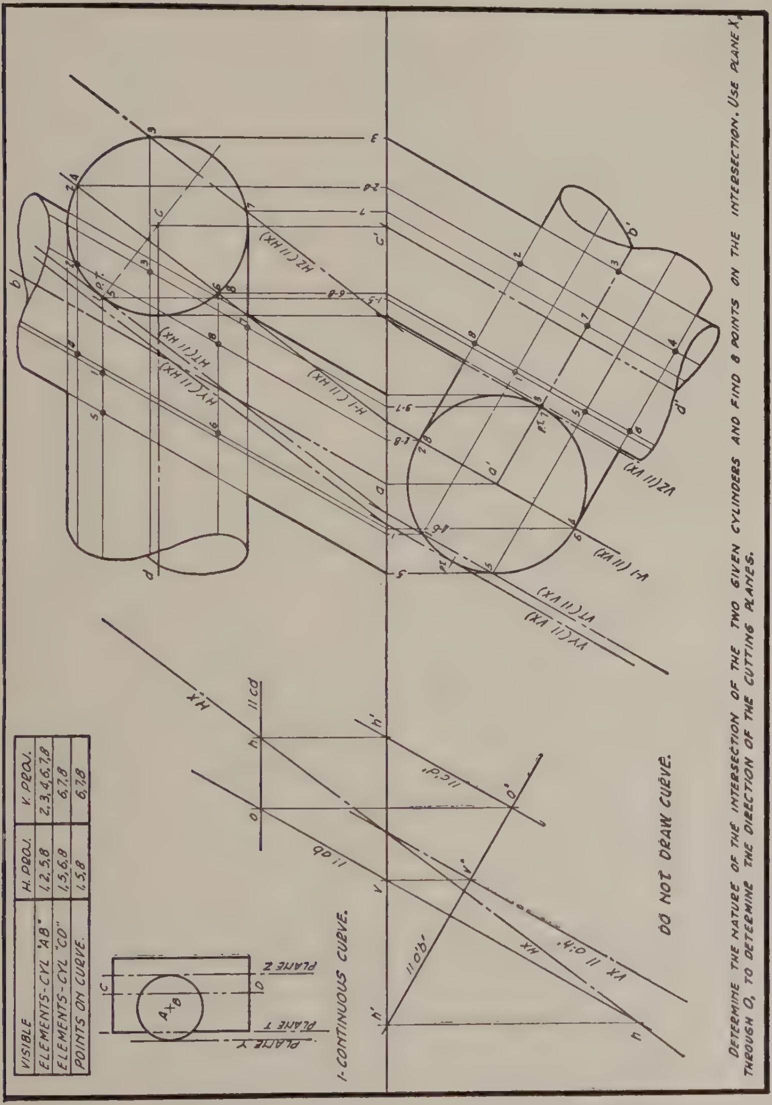
USE LIMITING PLANES X AND Z AND ONE GENERAL CUTTING PLANE T. (T-CHOSEN BECAUSE CUTS OUTLINE ELEMENT 4-6 FROM CYLINDER IN V-PROJECTION.)

DO NOT DRAW CURVE.

DETERMINE THE NATURE OF THE INTERSECTION OF THE GIVEN CYLINDER AND CONE AND FIND 8 POINTS ON THEIR LINE OF INTERSECTION.

Fig. 9.

at all, it is the arrangement of Fig. 8, in which there is but one continuous curve of intersection. These facts can and should be



1-CONTINUOUS CURVE.

DO NOT DRAW CURVE.

DETERMINE THE NATURE OF THE INTERSECTION OF THE TWO GIVEN CYLINDERS AND FIND 8 POINTS ON THE INTERSECTION. USE PLANE X, THROUGH O, TO DETERMINE THE DIRECTION OF THE CUTTING PLANES.

FIG. 10.

determined as the first step in the solution of such problems. The details of construction are shown in Figs. 9 and 10.

**128.** Problem 34. **To find the line of intersection of two given cylinders.**

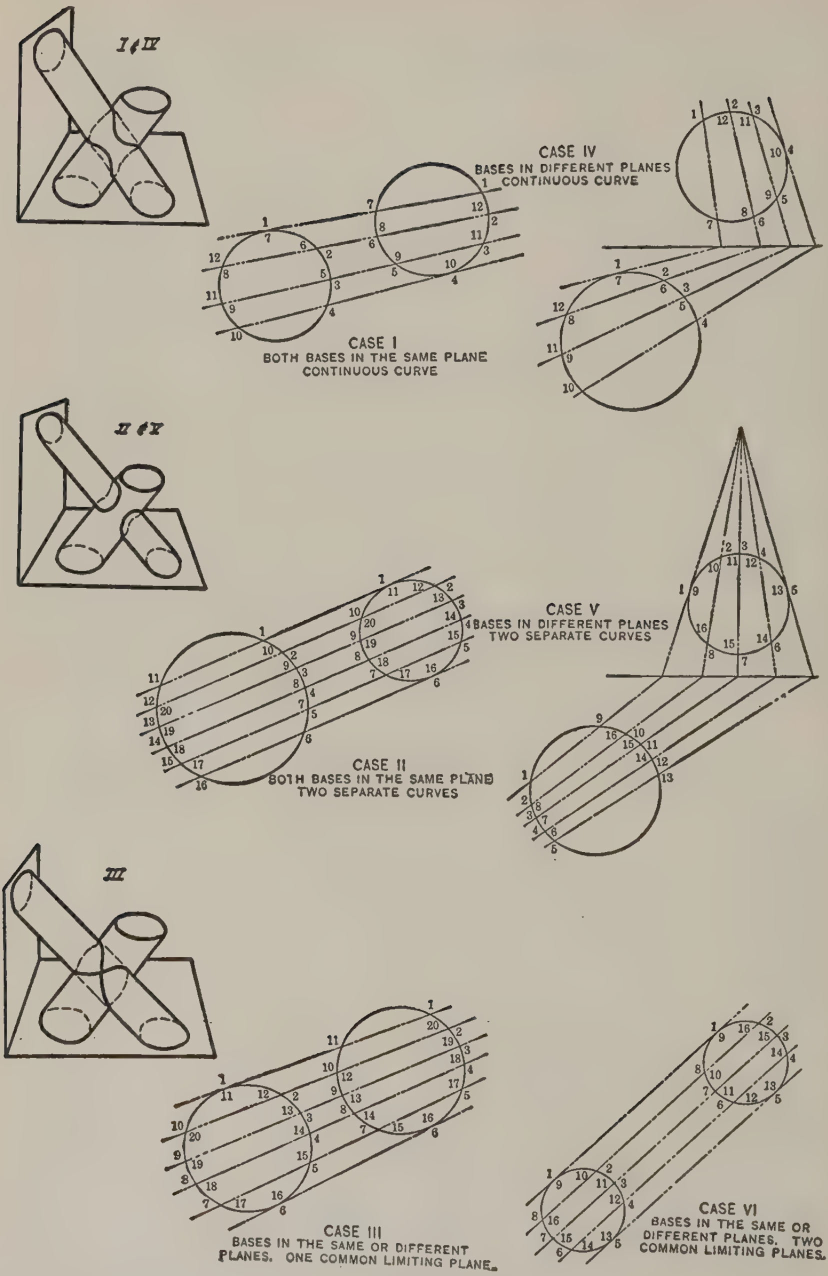

I & IV
CASE IV
BASES IN DIFFERENT PLANES
CONTINUOUS CURVE
CASE I
BOTH BASES IN THE SAME PLANE
CONTINUOUS CURVE
II & V
CASE V
BASES IN DIFFERENT PLANES
TWO SEPARATE CURVES
CASE II
BOTH BASES IN THE SAME PLANE
TWO SEPARATE CURVES
III
CASE III
BASES IN THE SAME OR DIFFERENT PLANES. ONE COMMON LIMITING PLANE.
CASE VI
BASES IN THE SAME OR DIFFERENT PLANES. TWO COMMON LIMITING PLANES.

FIG. 11.

**Analysis:**

The line or lines of intersection of the given cylinders are defined by a series of points common to both surfaces.

Points are derived as intersections of lines which in turn define planes.

Points will be derived at the desired intervals to define the required line of intersection by passing a series of planes such that they will intersect the surfaces of both cylinders in straight lines. Such planes must of course be parallel to the elements of both cylinders.

**Solution:**

Common secant planes parallel to the elements of the two given cylinders intersect the given cylinders in elements which intersect in points on the required line of intersection. Test planes parallel to the secant planes and tangent to the opposite sides of either cylinder determine if the intersection is a single curve, or two separate curves with or without a common point.

**Construction** (*a*):

Given: The two cylinders, Fig. 12, having circular sections as shown in the horizontal plane *M*.

Required: The line or lines of intersection.

Point *O* (*1,2*) is chosen in any convenient position and through it are drawn lines *OA* and *OB* parallel to the elements of the two cylinders (*3,6*). Plane *AOB*, or *X*, intersects any horizontal plane indicated by *GL* in a line *HX* (*7,10*). The auxiliary secant planes will be parallel to plane *X*, and their intersection with the horizontal plane *M* in which the given sections of the cylinders lie will be parallel to *HX*.

Test planes *Y* and *Z*, tangent to the opposite sides of the smaller cylinder (*11,12*), indicate that this is Case I, Fig. 11, in which each cylinder cuts through the side of the other, producing a single line of intersection.

Planes *R* and *Y* (*11,13*) are the limiting secant planes which will secure points (two each) on the line of intersection, since each is tangent to one of the two cylinders and secant to the other.

Plane *S* is a typical secant plane intersecting each cylinder in two elements (*15,20*) and (*21,26*), which intersect each other in four points, *A*, *B*, *C*, and *D*, of the required line of intersection.

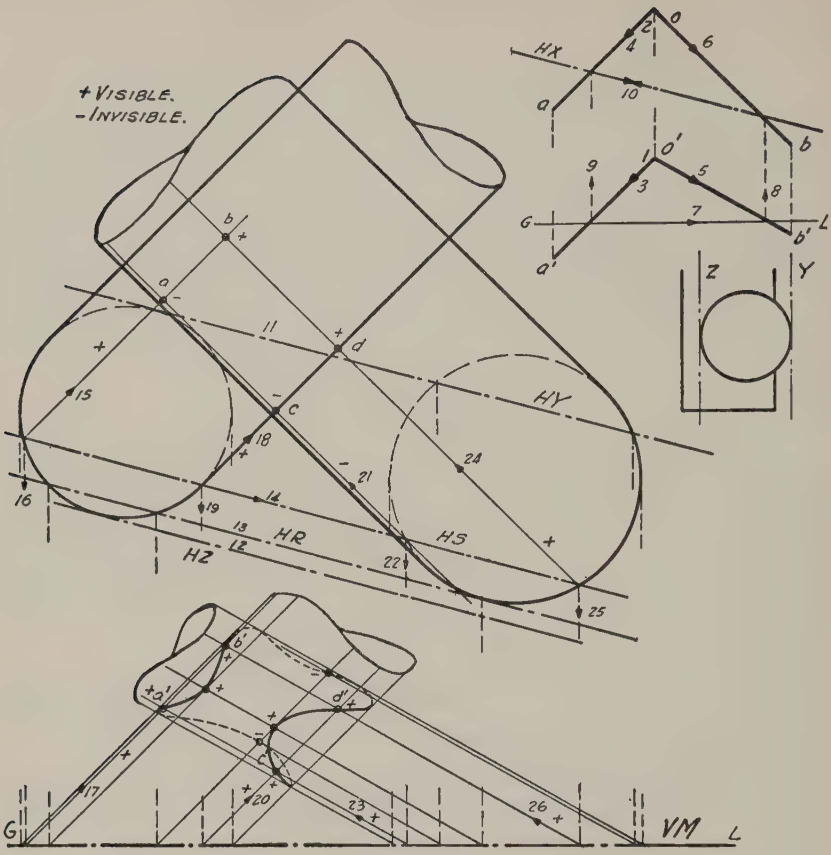


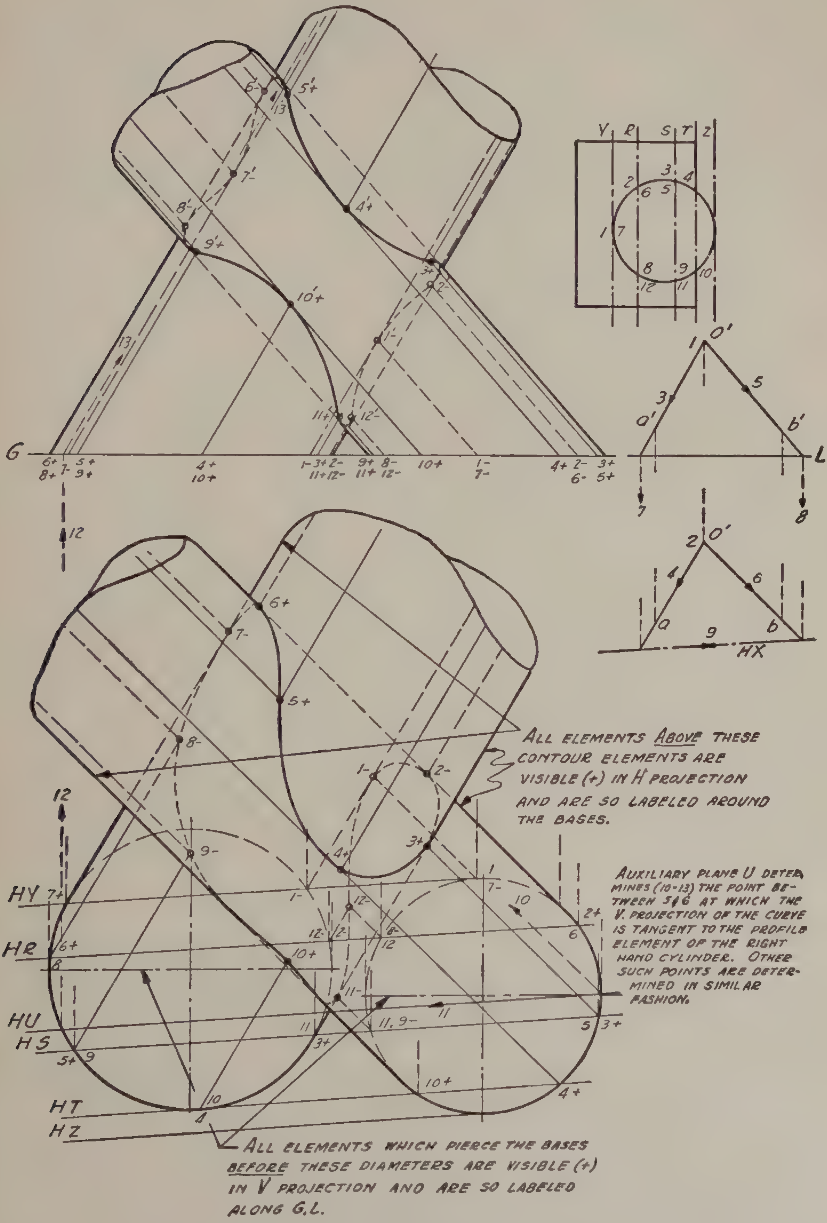
FIG. 12.

**Construction (*b*):**

Given: Two cylinders, Fig. 13.

Required: The line or lines of intersection.

Through any auxiliary point *O* (*1,2*) two lines, *OA* and *OB*, are passed parallel to the elements of the two cylinders, respectively (*3,6*). The *H* trace of this plane is *HX* (*7,9*). The trace of each auxiliary plane is then passed parallel to *HX*. Test planes *Y* and *Z*, parallel to plane *X* and tangent to the opposite sides of the smaller cylinder, indicate that there is but one curve of intersection (Case I, Fig. 11). Three additional planes, *R*, *S*, and *T*, intersect each surface in two elements, except plane *T*, which intersects one cylinder in two elements and is tangent to the other.



ALL ELEMENTS ABOVE THESE CONTOUR ELEMENTS ARE VISIBLE (+) IN H PROJECTION AND ARE SO LABELED AROUND THE BASES.

AUXILIARY PLANE U DETERMINES (10-13) THE POINT BETWEEN 5 & 6 AT WHICH THE V. PROJECTION OF THE CURVE IS TANGENT TO THE PROFILE ELEMENT OF THE RIGHT HAND CYLINDER. OTHER SUCH POINTS ARE DETERMINED IN SIMILAR FASHION.

ALL ELEMENTS WHICH PIERCE THE BASES BEFORE THESE DIAMETERS ARE VISIBLE (+) IN V PROJECTION AND ARE SO LABELED ALONG G.L.

FIG. 13.

The method of numbering the elements in such fashion that the intersection of an element numbered 1 on the one cylinder with an element numbered 1 on the other may give a corresponding point numbered 1 on the line of intersection is illustrated under Case I, Fig. 11. The elements of the right-hand cylinder, Fig. 13, are numbered to correspond to the left-hand base of Case I, and the left-hand base of Fig. 13 corresponds to the right-hand base of Case I.

From each of the numbered positions on the bases, corresponding elements 1 and 1, 2 and 2, etc., are drawn to intersect each other in points 1, 2, etc. This is performed for the *H* projections and then for the *V* projections. The projections of the line of intersection are then drawn through the points so determined and in order of their numbers; that is, 1 to 2 to 3, etc.

Since the two surfaces are given in the first quadrant, the elements of each are visible (+) or invisible (−) as indicated for the *H* and *V* projections. If either of the two elements which intersect to produce a point is invisible (−) when viewed toward the plane in question, *H* or *V*, the corresponding projection of the point is marked invisible (−) as point 1 in both *H* and *V* projection. If both elements are visible in *H* projection, the *H* projection of their intersection is marked visible (+), as point 4 in both *H* and *V* projection. When all points of the two projections of the line of intersection are so labeled, the visible portions of the curves may be indicated by full lines and the invisible by broken lines.

**Construction** (*c*):

The combination of cylinders shown in Fig. 14 is common in engineering, and their intersection and developments can be secured most conveniently if the axis of one is drawn perpendicular to a coordinate plane, e.g., *H*, and the axis of the other parallel to *V*. Thus both axes are parallel to *V* and one perpendicular to *H*. The right section of each is assumed circular in Fig. 14. The procedure is the same, however, whether the sections are circular or elliptical.

A plane *R* is passed perpendicular to the axis of the oblique cylinder, intersecting the surface in a right section, which by hypothesis is a circle. If this circle is revolved into or parallel to one coordinate plane, any number of points may be located on it. *DE* is the axis of such a cylinder. Since the cylinder has been placed parallel to *V*. the circle may be revolved about the diameter *1–7*, which is parallel to the *V* plane, until the circle is parallel to *V*. The point 3, whose *V* projection is $3'$, is revolved to $3_r$, and the distance from *3* to the diameter *1–7* is the length of the line seen in revolved position as $3_r3'$. This line in its true position is parallel to *H* and perpendicular to *1–7*. The *H* projection of *3* is therefore this same distance from the *H* projection of the axis *DE*.

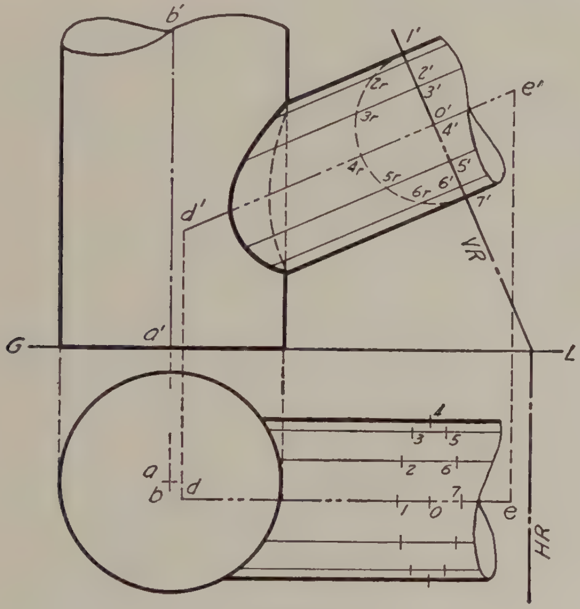


Fig. 14.

**129.** Problem 35. **To find the line of intersection of a given cone with a given cylinder.**

**Analysis:**

The line or lines of intersection of the cylinder and cone are defined by a series of points common to the two surfaces.

The required points will be derived by passing a series of secant planes such that they will intersect the cone and cylinder in straight lines.

**Solution:**

Common secant planes through the apex of the cone and parallel to the elements of the cylinder intersect both surfaces in elements which intersect in points on the required line of intersection. Two such secant planes tangent to the cone determine the number of curves and their nature.

**Construction:**

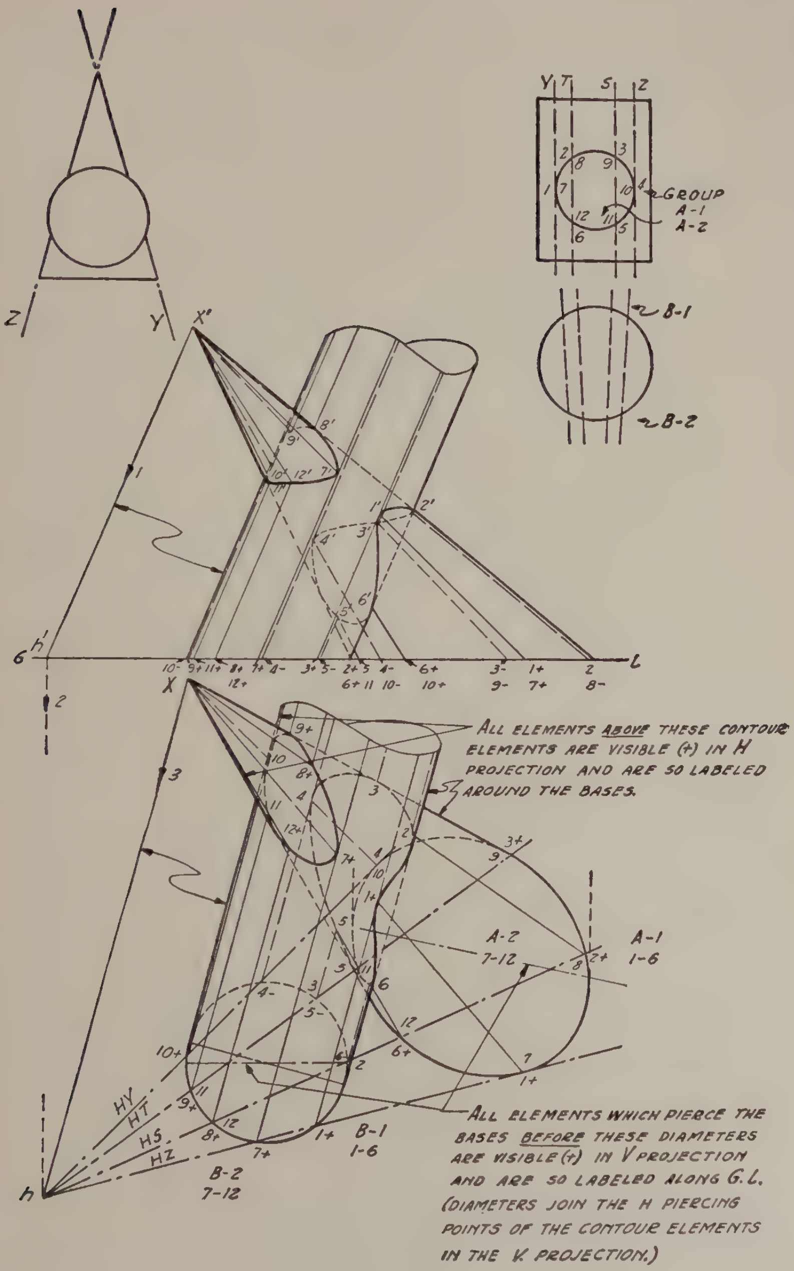
Given: Cone with apex at $X$ and cylinder, Fig. 15.

Required: Line or lines of intersection.

Line $XH$ is passed through the apex of the cone and parallel to the elements of the cylinder (*1,3*). $H$ is the point in which the line pierces the plane of the given sections or bases. The lines of intersection of all secant planes with the plane of the bases therefore pass through this point $H$.

Test planes $Y$ and $Z$, which are in this case tangent to the cone on opposite sides, indicate that there are two separate lines of intersection, Case II, Fig. 11. The fact that each plane has but one line in common with the cone but cuts two elements from the cylinder indicates that the cylinder is outside of the cone on both sides.

Either one of the test planes, $Y$ or $Z$, provides but one point on each line of intersection, since it cuts but one element from the cone to intersect each of the two elements cut from the cylinder. However, each additional plane, such as $S$ or $T$, gives two points on each line of intersection, since each of the elements it cuts from either surface will intersect both of the elements cut from the other surface. The method of numbering the elements in such fashion that one group of numbers shall apply to one line of intersection and another distinctly different group shall apply to the other is shown on the space diagram accompanying Fig. 15 and on the diagram describing Case II, Fig. 11. Since the cone enters the cylinder on one side and emerges on the other, each of the six elements of the cone will produce a point of the first curve. They may be considered Group *A–1* and numbered *1* to *6*, Fig. 15, or *1* to *10*, Case II, Fig. 11, where six planes have been used. And for convenient reference the numbers of this group should be placed outside the circle. Then the group of elements, *B–1*, of the cylinder which these will intersect first needs to be numbered in the same order from *1* to *6* to correspond.



ALL ELEMENTS ABOVE THESE CONTOUR ELEMENTS ARE VISIBLE (+) IN H PROJECTION AND ARE SO LABELED AROUND THE BASES.

ALL ELEMENTS WHICH PIERCE THE BASES BEFORE THESE DIAMETERS ARE VISIBLE (+) IN V PROJECTION AND ARE SO LABELED ALONG G. L. (DIAMETERS JOIN THE H PIERCING POINTS OF THE CONTOUR ELEMENTS IN THE V. PROJECTION.)

FIG. 15.

The same six elements of the cone serve as group *A–2* and are numbered *7* to *12*. They emerge on the other side of the cylinder, where they intersect group *B–2* of the cylinder numbered similarly *7* to *12*.

With the elements so numbered, six points on each of the two lines of intersection are secured. That is, the *H* projection of element *1* of Group *A–1* intersects the *H* projection of element *1* of Group *B–1* in the *H* projection of point *1*. The *V* projections of the same elements intersect in the *V* projection of point *1*.

Since the two surfaces are given in the first quadrant, the elements are visible or invisible as indicated by the signs + and −. Thus in *H* projection element *1* (+) of the cone intersects element *1* (+) of the cylinder in point *1* (+) of one line of intersection. This is the only visible point on this line in *H* projection because in each other point one of the two elements is invisible (−). Four points (*7*,*8*,*9*, and *12*) of the second line are visible in *H* projection, however, because the elements of these numbers of both surfaces are visible.

**130.** Problem 36. **To find the line of intersection of two given cones.**

**Analysis:**

The line or lines of intersection of the two cones are defined by a series of points common to the two surfaces.

The required points will be derived by passing a series of secant planes such that they will intersect both cones in straight lines. Such planes must pass through both apexes.

**Solution:**

Common secant planes through the apexes of the given cones intersect the cones in elements which intersect in points on the required line of intersection. Tangent planes will determine the number and relation of the curves.

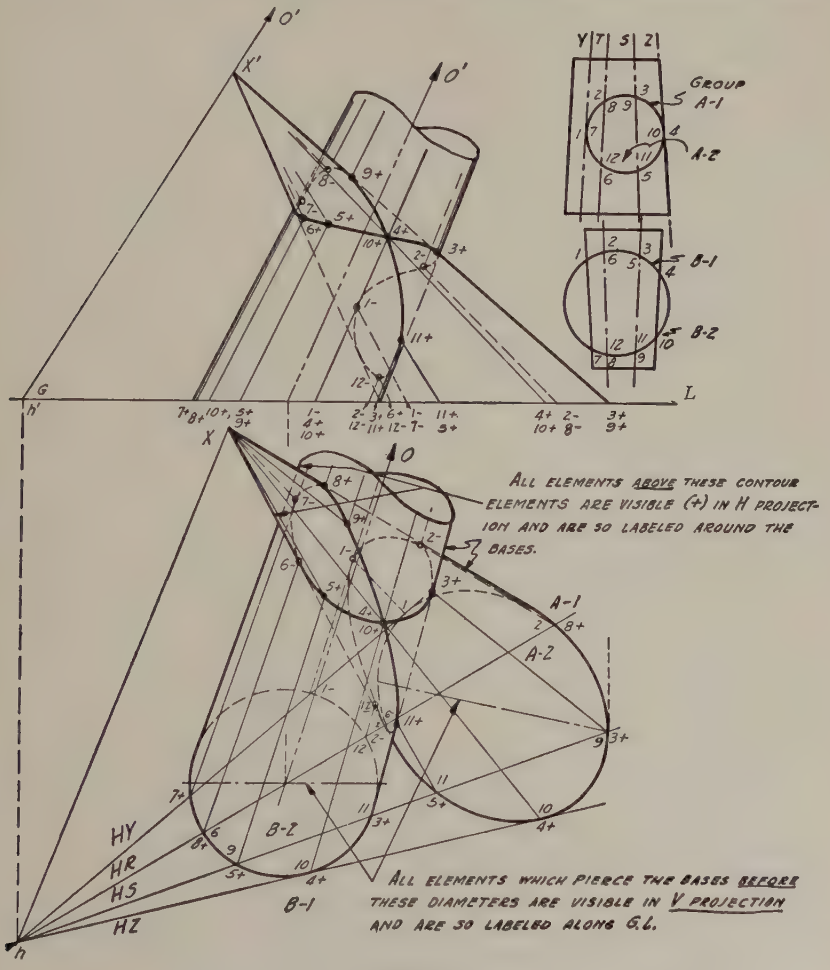
**Construction:**

Given: Cones *X* and *O*, Fig. 16.

Required: The line or lines of intersection.

The two cones of Fig. 16 have been made to resemble the cone and cylinder of Fig. 15 so closely that the processes of solution can be seen to be almost identical. The initial line through apex

*X* must of course pass through apex *O* also. All planes through this line will intersect the surfaces in elements if at all. Test planes *Y* and *Z* through *XO* indicate two separate lines of intersection as in the previous problem, but with a common point,



All elements above these contour elements are visible (+) in H projection and are so labeled around the bases.

All elements which pierce the bases before these diameters are visible in V projection and are so labeled along G.L.

Fig. 16.

Case III, Fig. 11. Two additional planes, *R* and *S*, give a total of six elements of cone *X* intersecting seven elements of cone *O*, numbered and divided into groups *A–1*, *A–2*, *B–1*, and *B–2* as shown.

The procedure of finding the intersections of elements *1*, *2*, *3*, etc., of Group *A–1* with elements *1*, *2*, *3*, etc., of Group *B–1* is the same as for the previous two problems. The visibility of the points so defined is controlled by the visibility of the pairs of elements, *1* and *1*, *2* and *2*, in *H* projection and in *V* projection respectively. The determination of the visibility of the various elements is indicated on the figure. That is, elements *2/8*, *3/9*, *4/10*, and *5/11* of cone *X* are visible (+) in *H* projection because they are *above* the profile elements. Elements *3/9*, *4/10*, *5/11*, and *6/12* of cone *X* are visible (+) in *V* projection because they pierce the plane of the base *before* the diameter through the *H* piercing points of the contour elements in *V* projection. Point *4/10* is the common point produced by the elements of tangency of plane *Z* with the two cones.

**131.** Problem 36: Example 1. **To pass a line through a given point making given angles $\theta$ and $\phi$ with *H* and *V*.**

**Analysis:**

All elements of a $\theta$ cone whose base angle is equal to the given angle $\theta$ and whose apex is the given point will pass through the given point and will make with *H* an angle equal to $\theta$.

All elements of a $\phi$ cone whose base angle is equal to the given angle $\phi$ and whose apex is the given point will pass through the given point and will make with *V* an angle equal to $\phi$.

If two cones which have a common apex intersect, the intersections are elements which have the properties of the elements of both cones.

If the elements of the above cones are made equal, their bases will lie on the surface of a sphere whose center is their apex and radius equal to the common length of elements.

**Solution:**

If the given point is made the center of a sphere of any convenient radius and the common apex of $\theta$ and $\phi$ cones whose elements are equal to the radius of the sphere, the bases of the cones lying on the surface of the sphere will intersect in two points which with the common apex will determine the intersections of the two cones, hence the two desired lines.

**Construction:**

Given: Point *O*, $\theta = 45°$, $\phi = 30°$, Fig. 17.

Required: Line through *O* at $\theta$ and $\phi$ with *H* and *V*.

Point $O$ is made the center of a sphere of 1 inch radius (*1,2*). The projections of a 45 degree $\theta$ cone with apex at $O$ and base on the surface of this sphere are then constructed (*3,7*). The projections of a 30 degree $\phi$ cone with apex at $O$ and base on the surface of the sphere are constructed (*8,11*). The $V$ projections of the bases of the two cones intersect at $c'$ and $d'$ and the $H$ projections at $c$ and $d$. The intersections of the two cones and the required lines through $O$ are $OC$ and $OD$ (*12,15*).

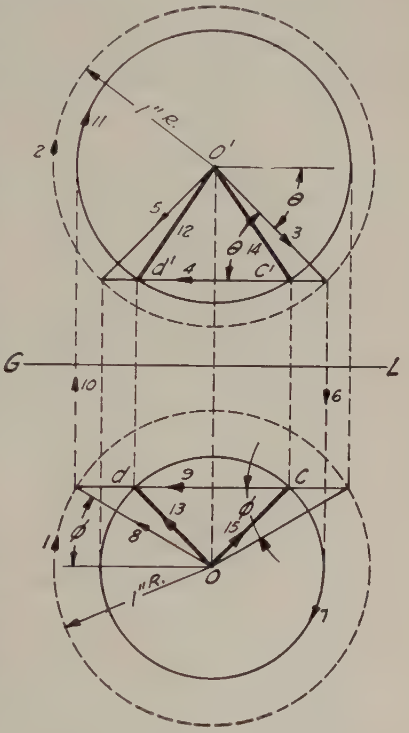


FIG. 17.

132. Problem 36: Example 2. **To pass through a given point a plane which will make given angles θ and φ with $H$ and $V$.**

**Analysis:**

Any plane $T$ which is perpendicular to a line $AB$ makes with any other plane, $H$, $V$, $P$, or $S$, an angle equal to the complement of the angle which the line makes with the same plane.

If it is possible then to construct through a given point $O$ a line which makes with any given plane angles which are complementary to the angles which it is desired that the required plane make with the given planes, a plane through the given point and perpendicular to this line must be the required plane.

See Article 131 for analysis and demonstration of the problem of passing through a given point a line which will make specified angles with a given plane.

**Solution:**

If through the given point or any chosen point, Fig. 18(*a*), a line is passed making with $H$ and $V$ complementary angles to the specified angles $\theta$ and $\phi$ for the plane, a plane through the point and perpendicular to this line will be the required plane.

**Construction:**

Given: Point $O$ and angles $\theta = 60°$ and $\phi = 45°$, Fig. 18(*b*).

Required: A plane $T$ through $O$ at specified angles of $\theta$ and $\phi$ with $H$ and $V$.

$$\left.\begin{array}{l} 90° - \theta\ (60°) = 30° = \theta \\ 90° - \phi\ (45°) = 45° = \phi \end{array}\right\}\text{Line}$$

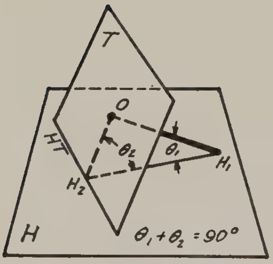


Fig. 18(*a*).

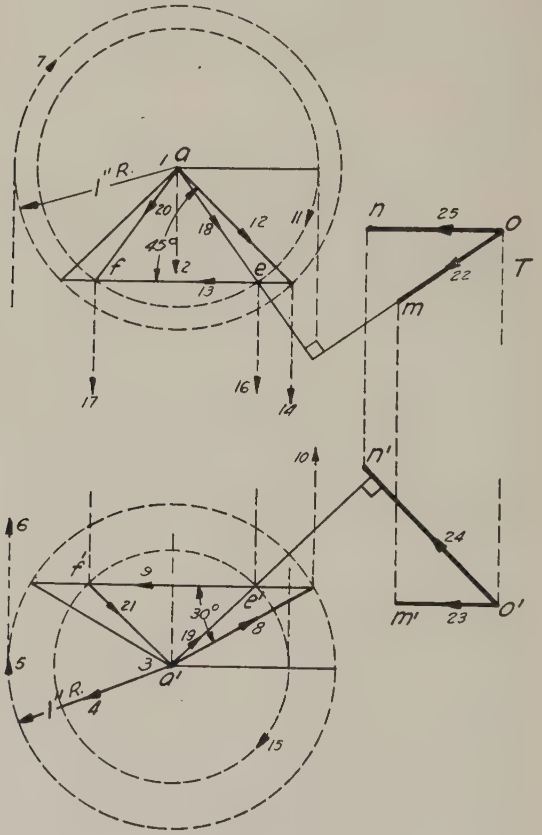


Fig. 18(*b*).

Any point $A$ is chosen as the center of a sphere of any radius (*1,7*). $A$ is made the apex of a $\theta$ cone of base angle of 30 degrees and with elements equal to the radius of the sphere (*8,11*). The base of this cone lies on the surface of the sphere. $A$ is made the apex also of a $\phi$ cone of base angle of 45 degrees and elements equal in length to the radius of the sphere (*12,15*). The base of this cone must also lie on the surface of the sphere. The two bases intersect at $E$ and $F$ (*16,17*), and the two elements of intersection of the cones are $EA$ and $FA$ (*18,20*). These lines make with $H$ and $V$ the complements of the given angles. Plane $MON$ through point $O$ perpendicular to $EA$ (*22,25*) is therefore one plane through $O$ at the specified angles with $H$ and $V$. Another plane would be perpendicular to $AF$.

**133.** Problem 36. Example 3. **To find the piercing points of a given space curve in the surface of a given cone or cylinder.**

**Solution:**

If the intersection of the $H$ or $V$ projecting cylinder of the given curve and the given cylinder or cone is determined, the intersections of the given curve and this line of intersection are the required piercing points.

## CHAPTER X

# DEVELOPMENT OF CYLINDERS AND CONES

**134.** The development of a cylinder or cone is the surface of that body rolled out on a plane; that is, a plane is passed tangent to the cylinder or cone through one of its elements and the surface is then split along any element and rolled out on this tangent plane. Any cylindrical or conical surface can thus be made from a flat sheet of metal.

The layout of the fluid conduit shown in Fig. 1 is made up of at least twenty-two separate pieces comprising sections of cylinders and cones. Each was made from a flat sheet of metal on which all openings, rivet holes, and laps were accurately laid out, the openings cut out, and the rivet holes punched before any bending or forming was done. The metal used throughout may have been one-quarter inch thick, or heavier, and due allowance was made in the layout for the effects of this thickness on the interior or exterior dimensions of the formed section. The necessary provision for fitting one end of one section into the end of the next or against the side of another was likewise made.

It is not our purpose nor our function to deal here with any part of the problem except to make the layouts of the various sections with no provision for thickness or for joining and fitting. Those are matters which are dictated by thickness, quality, and nature of materials used, the sizes of piping and the methods of joining: that is, by double seaming, riveting, or welding.

The process of producing a plane equivalent of the surface of a cylinder or of a cone and any lines on those surfaces may appear simpler if the cone is regarded as satisfactorily approximated for engineering purposes by an inscribed pyramid of large number of faces, and the cylinder by an inscribed prism.

The number of faces in each case needs to be large in order that a development so worked out may satisfactorily approximate the desired curved surfaces and their openings. Develop-

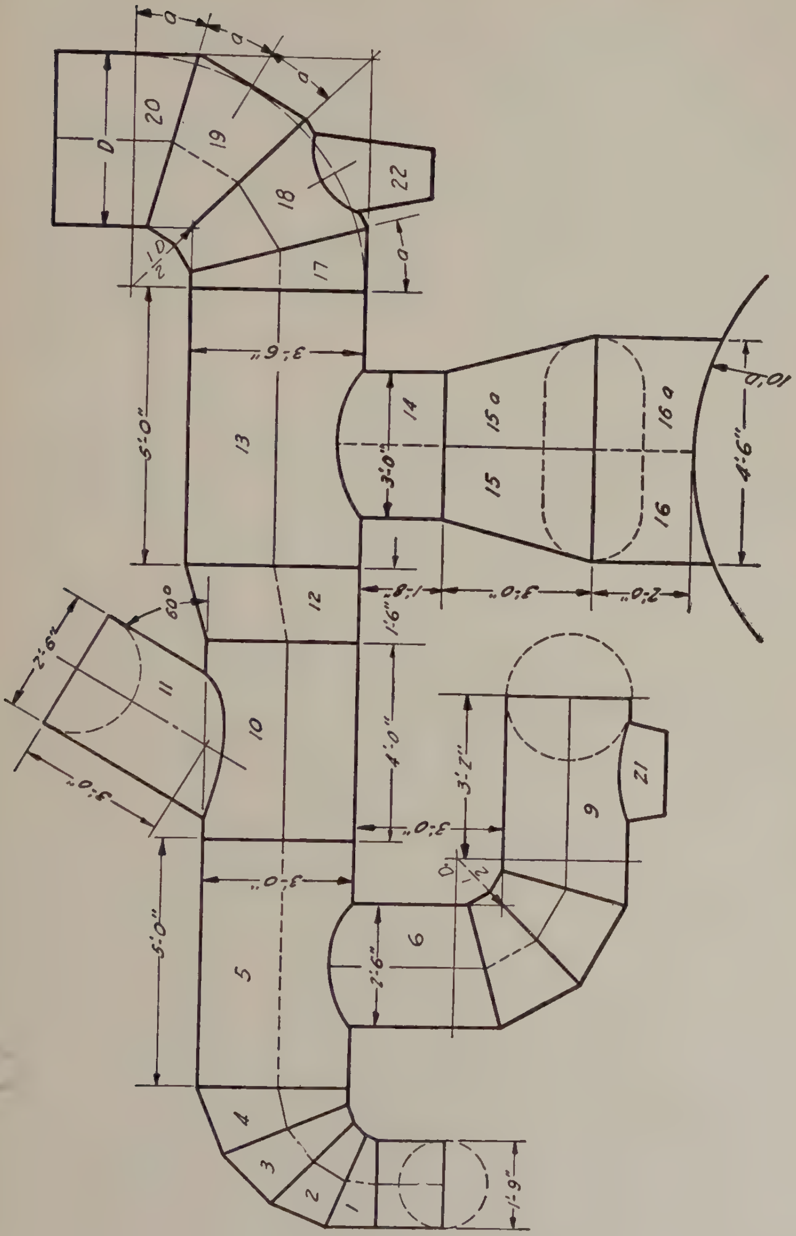

a
a
a
a
D
$\frac{1}{2}$D
20
19
18
17
22
3'-6"
5'-0"
13
14
3'-0"
15
15a
16
16a
4'-6"
10 D
1'-8"
3'-0"
2'-0"
1'-6"
12
60°
2'-6"
11
3'-0"
10
4'-0"
3'-2"
3'-0"
$\frac{1}{2}$D
21
9
3'-0"
5'-0"
5
6
2'-6"
4
3
2
1
1'-9"

FIG. 1.

ments are frequently worked out in two or more parts because of the sizes of sheets of metal required, difficulties in forming, and handling. It is best therefore to regard the cylinder or cone in terms of an inscribed prism or pyramid of a number of sides that is divisible by two, three, and four, that is, twelve or twenty-four. For large piping much larger numbers will be used, thirty-six, forty-eight, and sixty.

135. Problem 37. **To develop the surface of a cylinder and trace upon the development the intersection of the cylinder with any plane or another surface.**

**Analysis:**

If the cylinder is regarded in terms of an inscribed prism of at least twelve faces, the pattern produced by laying the faces of this prism with the respective portions of any lines of the surface side by side may for practical purposes be considered the development of the surface of the cylinder.

**Solution:**

If a series of elements of the cylinder is assumed at regular intervals and these are regarded as the edges of an inscribed prism, the development of this prism with the points of intersection of the given curve and the elements or edges is approximately the required development.

**Construction** (*a*):

Given: Right cylinder and the intersection of its surface with plane *T*, Fig. 2.

Required: The development of the surface and the line of intersection.

The circumference of the base is divided into twelve equal parts and elements passed through these points. The chord of one of the twelve arcs of the base is laid off twelve times on a horizontal line *1–12–1* as the development of the circumference of the base. Vertical lines at each of these points and with a common height of that of the cylinder gives the developments of the various elements. A horizontal line joining their tops is the development of the upper base.

The line of intersection starts on the lower base at a distance from *1* toward *2* equal to the chord "*a*" shown. The distances above the base at which the line intersects successive elements are shown in true value on the *V* plane because the lines are vertical. They are therefore projected horizontally to the proper element in the development. The curve is then drawn smoothly through these points. If in the finished pipe only that portion above the line of intersection is desired, the portion below the line may be regarded as cut out. In this fashion patterns for sections *5*, *9*, *10*, *11*, *13*, *14*, *16* and *16a*, *17* and *20* of Fig. 1 would be produced.

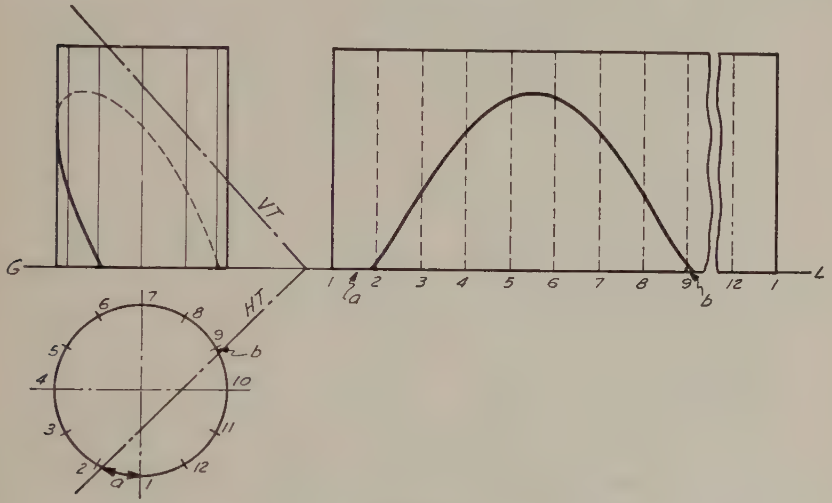


Fig. 2.

**Construction** (*b*):

Given: The oblique cylinder, Fig. 3.

Required: Development of cylinder showing line of intersection of cylinder and plane *T*.

The base section is divided into at least 12 equal arcs and elements are passed through their division points.

It is necessary here to pass a plane perpendicular to the elements of the cylinder so that it may intersect the cylinder in a right section which will develop as a straight line perpendicular to the elements. Plane *T* is then revolved either about *VT* into *V*, or about its intersection with the two contour elements

(*4* and *10*) of the cylinder until it is parallel to *V*, when the revolved right section will appear in its true size as shown. The true distances between these elements are shown in the revolved position of the right section. The development of the right section is the line *AB* as in the case of the right cylinder of Fig. 2. The several points of this line, *5*, *6*, *7*, etc., are determined by laying off distances equal to the chords of the revolved right section, and through these points the elements are drawn perpendicular to *AB*. The lengths of the various elements above and below the right section to the upper and lower bases show in their true value on the *V* plane or in elevation since they are parallel to *V*. These values may hence be projected parallel to *AB* to the developments of the corresponding elements. The curves drawn through the points so determined constitute the developments of the upper and lower bases. This construction illustrates the process of securing the developments or patterns for sections *6*, *7*, *8*, *18*, and *19* of Fig. 1.

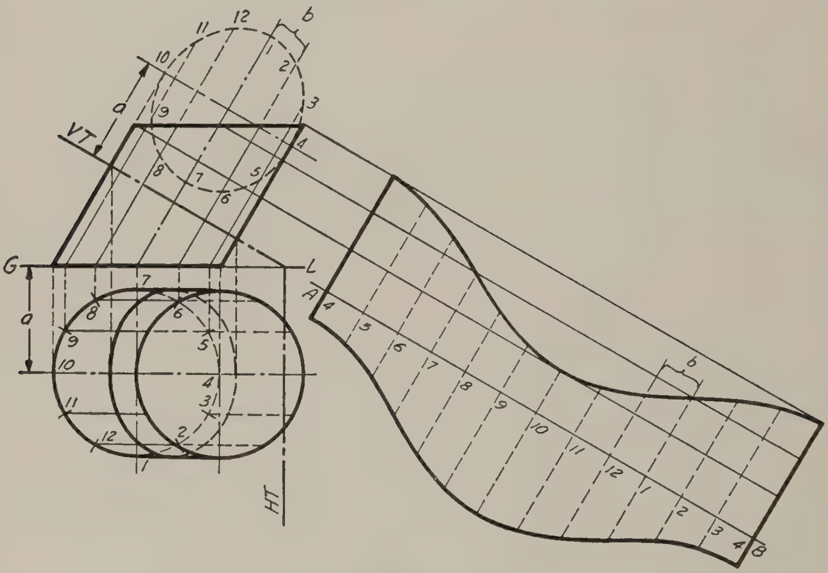


FIG. 3.

**136.** Problem 38. **To develop the surface of a right circular cone and trace upon the development the curve of intersection of the cone and any surface.**

**Analysis:**

If a pyramid of a sufficiently large number of faces is inscribed within the given cone, the development of this pyramid and of the points in which the curve intersects its edges will be the practical equivalent of the desired development of the surface of the cone and curve of intersection.

The faces of a regular pyramid are identical isosceles triangles whose equal sides in this case are equal to the elements of the cone. The development of the pyramid is hence a series of identical isosceles triangles with common vertex and sides equal to the length of the elements of the cone.

**Solution:**

If the base of the cone is divided into equal arcs—preferably twelve—and the elements are drawn to these points, they constitute the edges of the inscribed pyramid which may be developed as a series of adjacent equal triangles, each containing its portion of the curve of intersection.

**Construction:**

Given: Right circular cone $X$ intersected by plane $T$ perpendicular to $V$, and by horizontal planes $Y$ and $Z$, Fig. 4.

Required: The development of the portion of the surface between planes $Y$ and $T$.

The base of the cone in plane $Z$ is divided into twelve equal parts and the elements drawn. They intersect plane $T$ in points *1*, $A$, $B$, $C$, etc. Point $X_1$ is selected as the development of the apex and is made the center of an arc whose radius equals the length of the elements to plane $Z$, e.g., $x'1'$. From any point *1* chosen on the arc a chord equal to chord *1–2* is stepped off twelve times and the developments of the various elements *1* to *12* are drawn to the center $X_1$. The sector $1X_11$ then constitutes the development of the entire cone. A second arc is drawn with $X_1$ as center and $x'm'$ as radius. This constitutes the development of the surface to plane $Y$. The portion between $MM$ and *1,1* is the development of the surface between planes $Y$ and $Z$.

Plane $T$ intersects element *2* at $A$, *3* at $B$, etc. The true length of the element *3* from $X$ to $B$ is $x'b'_r$ by the process of rotating $XB$ around a vertical line until it is parallel to $V$ (*1,3*). This distance, $x'b'_r$, is laid off from $X_1$ on element *3*, giving $B_1$ as the development of point $B$ of the curve. Each point is so

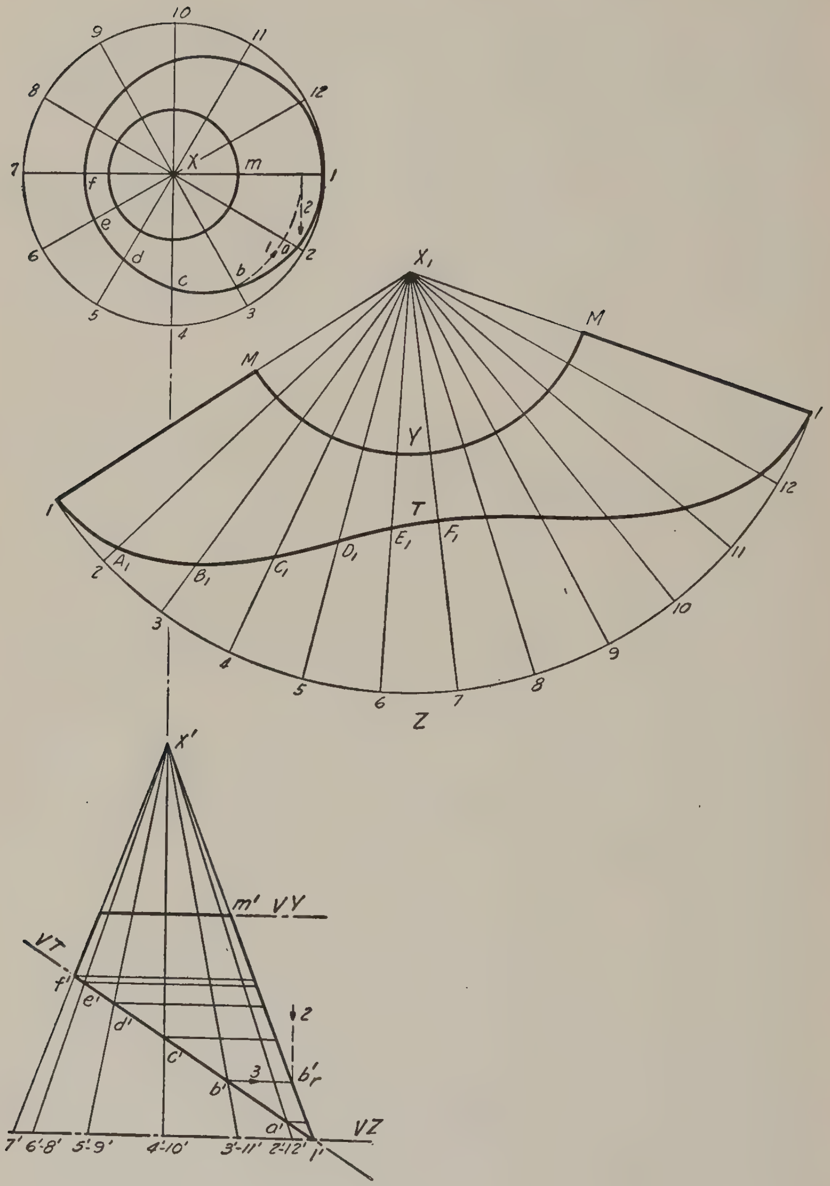

10
9
11
8
12
7
X
m
1
f
2
e
6
d
1
a
2
c
b
5
3
4
X₁
M
M
1
Y
12
1
T
F₁
E₁
D₁
A₁
C₁
B₁
11
2
3
10
4
9
5
8
6
7
Z
X'
m' VY
VT
f'
e'
d'
2
c'
3
b'ᵣ
b'
a'
VZ
7' 6'-8' 5'-9' 4'-10' 3'-11' 2'-12' 1'

FIG. 4.

determined and the development of the entire curve traced through the various points $A_1$, $B_1$, $C_1$, etc. The portion of the sector between arc $MM$, drawn with center $X$ and radius $x'm'$, and curve *1*, $A_1$, $B_1$, $C_1$, etc., is then the development of the portion of the surface of cone $X$ between planes $Y$ and $T$.

This process is employed to develop the patterns for sections 21 and 22 of Fig. 1.

**137.** Problem 39. **To develop the surface of an oblique cone and trace upon the development the line of intersection of the cone and any surface.**

**Analysis:**

If a pyramid of sufficiently large number of faces is inscribed within the cone, the development of this pyramid and of the points in which the curve intersects its edges will be the equivalent of the desired development of the surface of the cone and curve of intersection.

The faces of the inscribed pyramid will be irregular triangles, and it will be necessary to determine the lengths of the sides of each and fit the completed triangles together one by one.

If the surface is divided on the longest or shortest element, the development will be in identical halves, which will facilitate solution.

**Solution:**

If the base is divided into an even number of equal arcs, e.g., twelve, and the elements are drawn to these points, they constitute the edges of an inscribed oblique pyramid which may be developed as a series of adjacent irregular triangles, each containing its portion of the line of intersection.

**Construction:**

Given: Oblique cone $X$ intersected by horizontal planes $Y$ and $Z$, Fig. 5.

Required: Development of the portion of the surface of the cone between planes $Y$ and $Z$.

The base in plane $Z$ is divided into twelve equal arcs, beginning at the shortest element, and the elements are drawn. They intersect plane $Y$ in points which when developed will define the line of intersection with plane $Y$.

Point $x'$ is used for convenience as the development of the apex, $X_1$. Any convenient line through $X_1$ may represent the

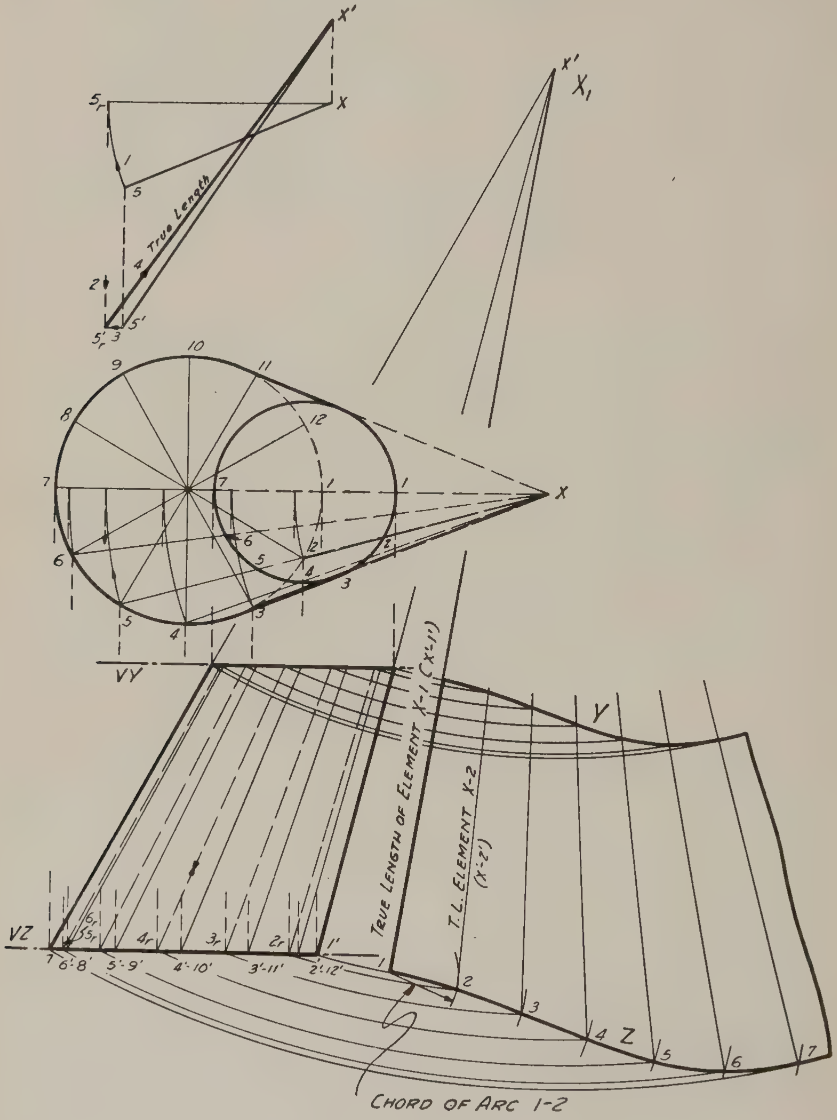


FIG. 5.

development of element *X1*, which shows in its true length as $x'1'$. The chord of the common arc *1–2* of the base is used as the common length base side of all the triangles. With development *1* as a center and the chord of the arc *1–2* as a radius, a short arc is described. The true length of element *X2* is then determined as in the illustration for element *X5*. With $x'$ as a center and the true length of *X2* ($x'2_r$) as a radius, the arc is described intersecting the short radius arc at point *2*. This process is repeated until the six triangles to the longest element are completed. This comprises half the development, which may be duplicated to produce the complete pattern.

As the true length of element *X2* was determined, the length of the portion of that element to plane *Y* was also determined and used as the radius of an arc to intersect the development of element *X2*. In this manner the development of each of the points in which the elements intersect plane *Y* is secured and the development of the curve is traced through the points so determined. The development between the upper and lower curves and from element *X1* to *X7* constitutes the development of half the surface of the oblique cone *X* between planes *Y* and *Z*. This process is employed to develop the patterns for the remaining sections of Fig. 1. These are numbers *1*, *2*, *3*, *4*, *12*, *15*, and *15a*.

# *PART III*

## Chapter XI

## SHADES AND SHADOWS

### SCOPE

**138.** In certain work of an architectural nature, it is frequently desirable to show the shades and shadows cast by the different parts of an object. It is not the purpose of this book, however, to treat the subject from any but a strictly mathematical standpoint, leaving the more elaborate discussions of degrees of shade, etc., to other works. Likewise only such problems will be included as are required for immediate and obvious needs.

### LIGHT—SOURCE—RAYS

**139.** Shades and shadows, indicating the absence of a certain amount of light from some surface, presume the *existence* and a *source* of light of some sort. For our present purposes the *sun* will be assumed as the *source*; and a shaft or beam of sunlight will be imagined as composed of countless *parallel rays*, which rays are *mathematical straight lines.*

### DIRECTION OF LIGHT

**140.** Upon the assumption of a source of light rays, the *direction* of these rays must be either determined or assumed. Shadows being used chiefly for effects, there can be no objection to the assumption of the direction of the rays. However, in assuming this direction, some care should be taken to make it approximately what it actually is for the *latitude* involved. A reasonably good rule is to assume both projections of the light rays to make with *GL* an angle equal to the *complement* of the *latitude* of the section of the country in question. The angle commonly used in the northern half of the United States is 45 degrees. As it is common practice to assume the light coming from the *left* and *rear*, the projections of the rays will

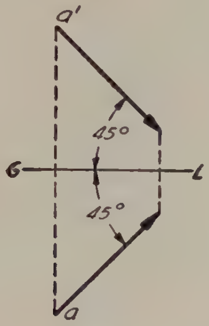


FIG. 1.

slope *toward GL* to the *right*, when objects are placed in the first quadrant, Fig. 1.

### DIRECT LIGHT—INDIRECT LIGHT

**141.** All light received from the source is known as **direct light,** and any portion of any surface from which the *direct light* is excluded by somebody is said to be **in shadow.** Most illumination, however, is *indirect* and in some cases it may be almost impossible to distinguish with the eye between the *shaded* and *directly illuminated* portions of a surface. There are always countless surfaces which reflect the direct light upon the shaded surfaces. This **reflected light** is known as **indirect light** and gives rise to **degrees of shade** mentioned before.

### SHADES—SHADOWS

**142.** Any portion of a surface of an object from which the *direct light* is excluded by some part of the *same* object is said to be **in shade,** or for convenience may be called **a shade.**

Any portion of a plane or the surface of an object from which the *direct* light is excluded by some part of *another* object is said to be **in shadow,** and for convenience may be called **a shadow.**

### LINE OF SHADE—SHADOW

**143.** The *line which separates the shaded from the illuminated portions* of any surface is known as the **line of shade,** as $BC$, Fig. 2. The *line which limits a shadow* on any surface is called a **line of shadow,** as $c_1b_1$, Fig. 2.

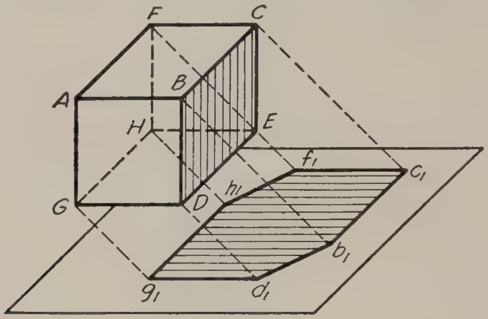


Fig. 2.

### SHADOW RAY

**144.** Much labor may be saved in finding the shadow of an object on a surface, if one determines by inspection what lines of the object will determine the limits of the shadow. It is easily seen in Fig. 2 that point $A$ will play no part in determining the shadow, while points $H$, $F$, $C$, $B$, $G$, and $D$ will. It is evident that by passing rays through $B$ and $C$ the points $b_1$ and $c_1$ are located; $b_1c_1$ is thus the shadow line determined by the edge $BC$. The line $Bb_1$ is in reality a ray of light, since it has been used in determining a point on the *line of shadow,* however, it will be known in this work as a **shadow ray.** *Any other such ray* which may be *used to determine any point* either *on a line of shade* or *on a line of shadow* will be known as a **shadow ray.**

### FIRST VERSUS THIRD QUADRANT

**145.** Shadows are usually rendered in elevation views only. Since the top view, or plan, does not have the shadows shown, it is advantageous to place the object in the *first,* rather than in the *third,* quadrant. By this arrangement the $V$ plane may be conceived as a wall of any building upon which shadows may be cast. Furthermore, it seems more natural to imagine the *H* plane to be the ground or plane of the base of the object than to use the extra plane when the object is in the third quadrant.

## SHADES AND SHADOWS—HOW REPRESENTED

**146.** Any given point can have but *one shade* or *one shadow.* Also, an object can be viewed from only *one point* at a time. Hence, when dealing only with the elevations or *V* projections of shadows cast upon any structure, the observer is *before V*, the orthogonal lines of sight are *perpendicular to V*, and as far as the observer is concerned there are *no shades or shadows on horizontal planes.* And unless so specified in the requirements of the problem, the projections of shades and shadows on *H* planes need not be found.

Since the subject of shades and shadows is being considered here solely in terms of orthogonal projection, it is seen that when observed from *before V* the *shade or shadow* of a given point *on a given oblique plane* is represented by the *V projection* of that shade or shadow; as far as the observer is concerned, an *H projection does not exist* and need not be found. When viewed from *above H*, as in a **bird's-eye view,** the *H projection* of the shade or shadow represents that shade or shadow and the *V projection* does not exist and need not be found.

**147.** Problem 40. **To find the shadow of a given point on a given surface.**

**Solution:**

If through the given point a shadow ray is passed, the point in which it pierces the surface is the required shadow.

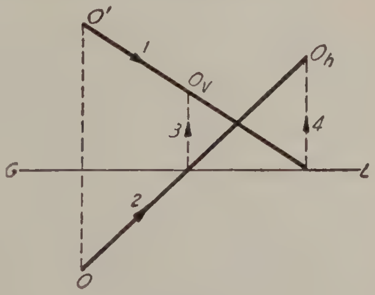


FIG. 3.

**Construction** (*a*):

Given: Point *O*, Fig. 3.

Required: Shadow of *O* on *V* or on *H*.

The projections of the shadow ray are drawn through *o′* and *o* at 45 degrees to *GL* (*1,2*); this ray pierces *V* at $O_v$ (*3*), the required shadow on *V*, or *H* at $O_h$ (*4*), the shadow on *H*.

**Construction** (*b*):

Given: Point *O* and plane *R*, Fig. 4.

Required: Shadow of *O* on *R*.

An *H* projecting plane is passed through the shadow ray from *O* (*1,3*); its line of intersection with *R* is *VH* (*4,5*). The shadow ray intersects this line at $O_1$, $o_1$, and $o_1'$ (*6*). If shadows are required in elevation, $o_1'$ is the desired shadow; in plan, $o_1$ is used.

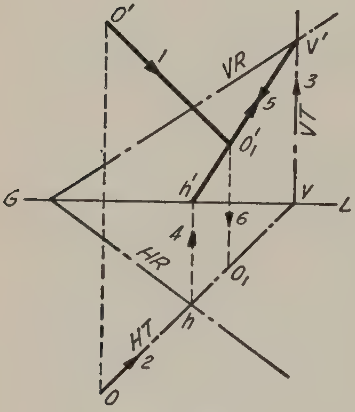


Fig. 4.

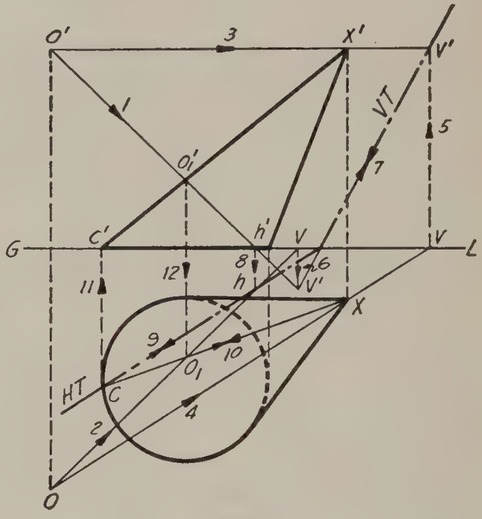


Fig. 5.

**Construction** (*c*):

Given: Point *O* and cone *X*, Fig. 5.

Required: Shadow of *O* on surface of cone.

The shadow ray is passed through *O* (*1,2*) and *O* is connected with the apex *X* (*3,4*). Plane *T* (*5,6*) of these two lines cuts the element *CX* (*10,11*) from the cone; the shadow ray intersects this element in $O_1$ (*12*). In dealing with shadows in elevation, $o_1'$ is the required shadow; in plan, $o_1$ is used.

**Construction** (*d*):

Given: Point *O* and the ellipsoid *X*, Fig. 6.

Required: Shadow of *O* on ellipsoid.

The *H* projecting plane *T* is passed through the shadow ray *OC* (*1,2*). Plane *T* cuts from the ellipsoid an ellipse whose minor axis is *aa*. The major axis *d′d′* is obtained by laying off the distance $e$ on the $V$ projection as shown (*5,9*). Plane $T$ is revolved parallel to $V$ about the axis of the ellipsoid as an axis of revolution (*10,15*). $OC$ comes into the position $O_2C_1$ (*16,17*), and the $V$ projection of the revolved position of the ellipse may be constructed in the position shown, with minor axis equal to $aa$ and major axis equal to $d'd'$; the ordinary methods of constructing an ellipse may be used. The revolved position of the required shadow is then found at $o_3'$ and the shadows themselves at $o_1'$ and $o_1$ respectively (*20,21*).

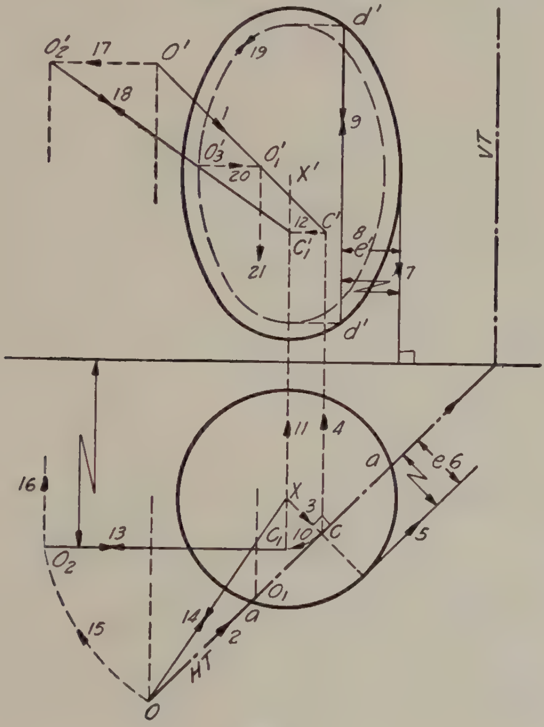


FIG. 6.

## BRILLIANT POINTS—LINES

**148.** If on a *surface of double curvature* there is a *point at which a ray of direct light is reflected to the eye, this point appears more brilliant* than any other point of the illuminated surface and is known as a **brilliant point.** On a *surface of single curvature* instead of a point there will be a *line along which direct light* *will be reflected to the eye* (if reflected at all); this line is known as the **brilliant line.**

Such *brilliant points* or *lines do not necessarily exist for every position of the point of sight and any assumed direction of light;* if the surface is either a sphere or an ellipsoid of revolution, however, such a brilliant point does exist and can be found.

149. Problem 41. **To find the brilliant point on the surface of a given ellipsoid of revolution.**

**Analysis:**

Assuming that it is desired to locate the brilliant point in elevation, the eye is an infinite distance before the $V$ plane; hence the reflected ray is perpendicular to $V$.

The bisector of the angle between the incident and reflected rays is normal to the surface of the ellipsoid at the brilliant point.

A normal to the surface of an ellipsoid passes through the axis of revolution.

**Solution:**

If through any convenient point in space a ray of light and a line of sight are passed (the latter perpendicular to $V$), the bisector of the angle between these two straight lines is parallel to the normal of the surface at the brilliant point. If then a plane is passed through the axis of the ellipsoid parallel to the bisector of this angle and a line is drawn in this plane perpendicular to the direction of the normal (or the bisector), and tangent to the ellipse of intersection of the plane with the ellipsoid, the point of tangency of this line with the ellipse is the required brilliant point.

**Construction:**

Given: The ellipsoid $XY$, Fig. 7.

Required: The brilliant point in elevation.

The ray of light $OB$ and the line of sight $OA$ are passed through any point $O$ (*1,4*). $OB$ is revolved about $OA$ as an axis until it is parallel to $H$ (*5,8*). $OM_r$ is the bisector of the revolved angle $AOB$, and its projections are $om$ and $o'm'$ (*9,13*). Plane $T$ is passed through the axis $XY$ of the ellipsoid parallel to $OM$; $HT$ (*14*) is parallel to $om$. Line $CD$ is passed through any point $C$ in $T$ parallel to $OM$; $CD$ lies in plane $T$. Plane $T$ is revolved about the axis $XY$ of the ellipsoid until it is parallel to $V$. The $V$ projection of the ellipse cut from the ellipsoid by $T$ now coincides with the $V$ projection of the ellipsoid, and the revolved position of $CD$ is described by the projections $c'_r d'_r$, $c_r d_r$ (17,23). The tangent line, $EF$, is determined by constructing the $V$ projection $e'_r f'_r$ (24) of its revolved position perpendicular to $c'_r d'_r$. The $V$ projection of the revolved position of the brilliant point is $p'_r$, and the projections of the required brilliant point are $p'$, $p$ (25,28).

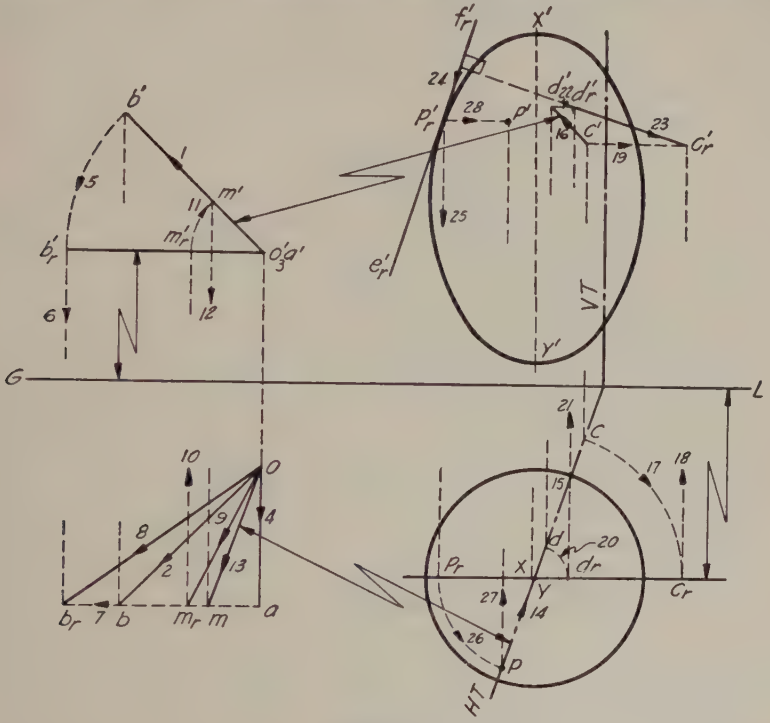


FIG. 7.

**150.** Problem 42. **To find the shade on a given ellipsoid of revolution and the shadow of the ellipsoid on $H$.**

**Solution:**

If a cylinder of light rays is passed tangent to the given ellipsoid, the line of tangency is the line of shade; the line of intersection with $H$ of this cylinder is the required line of shadow.

**Construction:**

Given: Ellipsoid $O$, Fig. 8.

Required: Shade and shadow of ellipsoid on $H$.

A light ray *AB* is passed through any convenient point *A* and is revolved about a vertical line through *A* until it is parallel to *V* (*1,6*). The ellipsoid and the cylinder of tangent rays are revolved about the axis of the ellipsoid until the elements of the cylinder are parallel to *V*. The upper and lower limiting

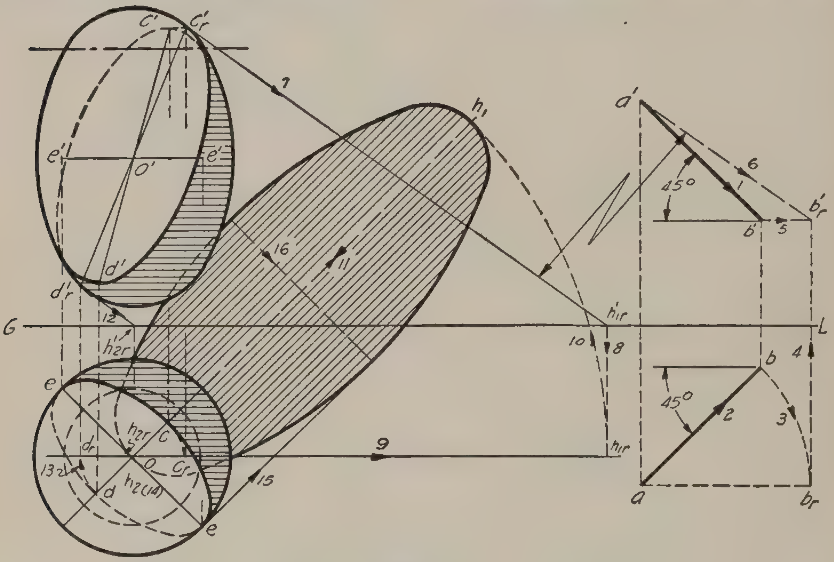


FIG. 8.

elements come into the positions $C_rH_{1r}$, $D_rH_{2r}$, the projections of which are $c'_rh'_{1r}$, $c_rh_{1r}$ and $d'_rh'_{2r}$, $d_rh_{2r}$, parallel to the projections of the revolved position of the light ray *AB*. The axes of the elliptical line of shadow on *H* are determined as equal to $h_{1r}h_{2r}$ (*major*), (*7,14*), and to the diameter of the circular *H* projection of the ellipsoid (*minor*), (*15,16*).

*CD* is the major axis of the elliptical line of shade. Its *H* projection *cd* becomes the minor axis of the elliptical *H* projection of the line of shade. The diameter *ee*, perpendicular to *cd* at its center, is the major axis. The normal positions of *C* and *D* (*cc'*, *dd'*) are determined from the revolved position $C_rD_r$ ($c_rc'_r$, $d_rd'_r$) by counter-revolution and provide two points each on the *V* and *H* projections. Other points on the *V* projection may be determined by cutting circles from the ellipsoid by auxiliary planes perpendicular to the axis, and finding the points of intersection of these circles with the *H* projection of the line of shade.

## CHAPTER XII

# PERSPECTIVE

### DEFINITION

**151.** A **perspective drawing** of any object is *such a representation as has the same appearance as the object itself when viewed from some definite point.* An object can be viewed from an infinite number of positions, each view differing slightly from the others. Any number of perspectives can therefore be made of the same object.

Perspective drawing is employed principally in architectural work. Orthogonal working drawings, composed of several views of an object, and made with a corresponding number of view-points at infinite distances from the coordinate planes present little but the bare details of construction. Isometric and oblique projections represent an object in a single view, but with all care and allowances such projections never appear quite natural.

It is then left to perspective drawing to present objects as they actually appear to us. While the chief use of perspective is in architectural drawing, the principles of construction, depending upon the principles of descriptive geometry, are easily understood and can be applied in a mechanical way by the engineer without any special artistic training.

### PICTURE PLANES

**152.** Perspective drawings are made from orthogonal working drawings; and, as working drawings are third quadrant projections, it will be assumed that in making a perspective the object is always placed in the third quadrant.

Any *plane parallel* to either the *V plane* or the *H plane* may be used as the **picture plane.** In this treatise, however, for simplicity of construction, the *V and H planes will be termed and* *used as the picture planes,* i.e., the planes on which the representations or perspective drawings are made.

*When the representation or drawing is made on the H plane,* it is termed a **bird's-eye view**; such drawings are made frequently of large areas of ground, cities, groups of buildings, etc. *If, however, the drawing is made on the V plane* it is termed a **perspective**; such drawings of course have the wider range of utility.

### POINT OF SIGHT—PRINCIPAL POINT

**153.** The *position of the eye of the observer* is known as the **point of sight** and is lettered $S$, Fig. 1. This *point of sight* is on *the opposite side of the picture plane from the object,* that the drawing may be smaller than the object; however, if it is desired to make the drawing larger than the object, a new picture plane may be assumed in such position as to bring both the object and the point of sight on the same side.

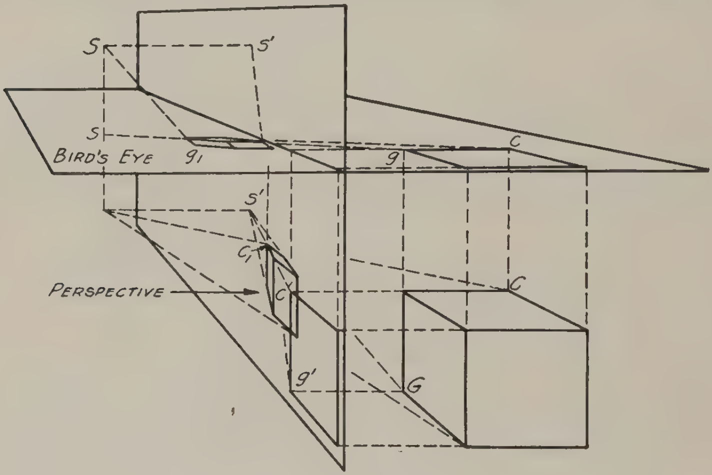


FIG. 1.

In making a *bird's-eye view* the *point of sight will be above the H plane* and, ordinarily, although not necessarily, in the first quadrant. For a perspective, $S$ is taken in *front of V and usually in the fourth quadrant.*

The $v$ projection of $S$ will be lettered $s'$, the $H$ projection $s$. $s'$ is of great importance in construction, hence is known as the **principal point**; its use will be explained later.

There is more to the locating of the point of sight with reference to the object than appears on first thought. The principles of construction can be applied as easily with $S$ in any one position as in any other; however, unless care is exercised, the drawings, though mathematically correct, may have a most unnatural appearance. One can imagine how absurd the perspective of a house would appear if made with the view point only a few feet from one of the walls. Suggestions for placing the point of sight will be given later.

## POSITION OF POINT OF SIGHT

**154.** Though no fixed rules for placing the point of sight will be given, a few suggestions will be helpful.

With beginners there is a tendency to simplify construction by placing the point of sight *too close* to the object, perhaps within the limits of the drawing. The result is invariably a "steep" perspective having a most unnatural appearance. *Too great a distance produces always less serious faults than too small a distance.* With large objects the point of sight should be before the picture plane a distance of at least three or four times the greatest dimension of the object.

*More natural perspectives* are produced with the *point of sight nearly in front* of the object. With the point too far to the side trouble is likewise experienced with too long intersections.

With small objects any rule but that of "good sense" is hard to give. The best solution of the matter is to ask one's self how far from the object should the eye of the observer be, and in just what position, to see all the details to the best advantage.

## VISUAL RAY—PLANE—CONE

**155.** A *line from the point of sight to any point of an object* is known as a **visual ray.** A *plane passing through the point of sight and any straight line of the object* is a **visual plane.** A *cone of visual rays with the apex at the point of sight* and the *surface of the cone tangent to a given curved surface* (sphere, etc.) *or all rays intersecting a given curved line* is known as a **visual cone.**

## PERSPECTIVE BY PIERCING POINTS

**156.** Problem 43. **To find the perspective of a given point.**

**Solution:**

The piercing point in $V$ (the picture plane) of a visual ray through the point of sight and the given point is the required perspective.

**Construction:**

Given: Point $O$, third quadrant, and point of sight $S$, Fig. 2.

Required: Perspective of $O$.

A visual ray is passed through points $S$ and $O$. Its projections are $s'o'$ and $so$ (*1*,*2*). This ray pierces $V$ at $V$, projections $v'$ and $v$ (*3*). The point $v'$ is the required perspective and will be lettered $o_1$ if lettered at all.

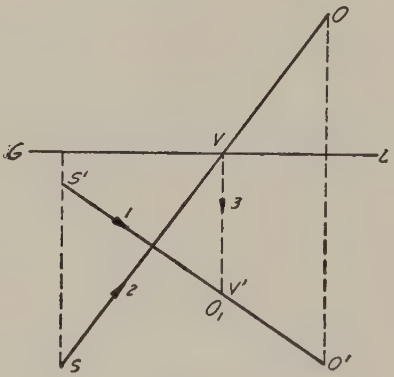


Fig. 2.

**Defect:** The theory of the problem just given is sufficient to construct completely the perspective of any object. In practice, however, the method can seldom be used; for the point of sight is generally in such position as to bring its $H$ projection $s$ without the limits of the drawing. This theory is, however, the basis of solutions to follow.

**157.** Problem 44. **To find the perspective of a given line.**

**Analysis:**

From the previous problem it is seen that, if a point lies in $V$, the piercing point of the visual ray and the given point coincide; i.e., the point is its own perspective. Then if a given line is produced until it pierces $V$, this point is one point of the perspective of the given line.

**Solution** ($a$):

The $V$ piercing point of the given line is one point of the required perspective. The $V$ piercing point of a visual ray to any other point of the given line is a second point of the required perspective.

**Construction:**

Given: Line $AB$, third quadrant, and point of sight $S$, Fig. 3.
Required: Perspective of $AB$.

$AB$, produced, pierces $V$ at $V$, which coincides with $v'$ (*1,2*). The visual ray from $S$ to $A$ pierces $V$ at $a_1$ (*3,5*). The perspective of $B$ lies on the $V$ projection $s'b'$ (*6*) of the visual ray from $S$ to $B$, and on the perspective $a_1v'$ of the line, hence at their intersection, $b_1$. $a_1b_1$ is the required perspective.

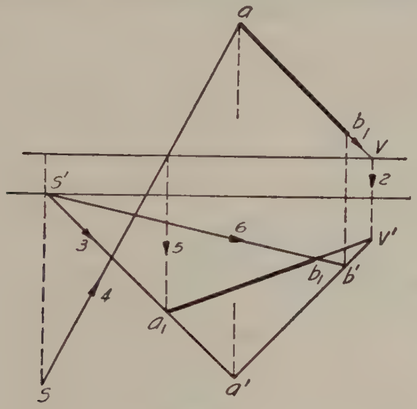


Fig. 3.

**Solution** ($b$):

The $V$ trace of a visual plane through the point of sight and the given line is the required perspective of the given line.

**Special lines.** From Solution ($b$) above, it is seen that: (1) *If a line is parallel to the picture plane* ($V$), *its perspective is parallel to the given line, likewise to the $V$ projection of the given line.* (2) *The perspective of a vertical line is a vertical line.*

**158.** Problem 45. **To find the vanishing point of a given group of parallel lines.**

**Analysis:**

If a group of parallel lines is produced infinitely, they appear to meet in a point known as their **vanishing point.** A visual ray parallel to the group of lines likewise appears to pass through their vanishing point. Its piercing point in *V* is therefore the perspective of this point. In perspective this piercing point is known as the vanishing point of the group of lines. The perspective of each line of the group passes through the vanishing point.

The vanishing point of a group of parallel lines is common to the perspectives of all lines of the group. It is necessary, therefore, to find the perspective of only one additional point of each line to determine the perspective of the line.

**Solution:**

The piercing point in *V* (the picture plane) of a visual ray through the point of sight parallel to the lines of the given group is the required vanishing point.

**Construction:**

Given: Point of sight and lines, *AB*, *CD* and *EF*, Fig. 4.
Required: Vanishing point of given lines.

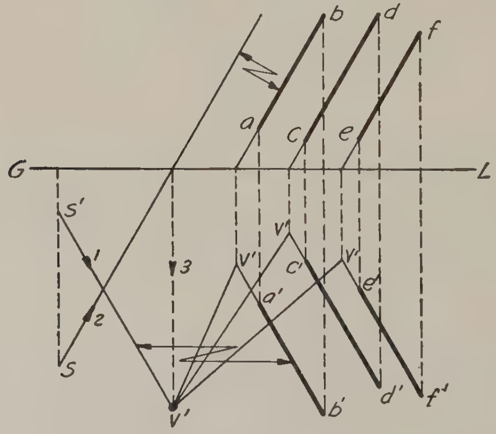


Fig. 4.

The projections of the ray through *S* parallel to the given lines are parallel to the respective projections of the given lines (*1*,*2*). The *V* projection *v'* (*3*) of its piercing point in *V* (the picture plane) is the required vanishing point.

## PERPENDICULARS—HORIZONTALS—DIAGONALS—POINTS OF DISTANCE

**159.** *Lines perpendicular to the V picture plane* are known as **perpendiculars;** *such lines are of course horizontal.*

If a group of perpendiculars is produced indefinitely, the perpendiculars appear to meet in a point on what we commonly call the **horizon.** *A visual ray parallel to these perpendiculars pierces V in s′*, the *V projection* of the *point of sight S.* Hence s′ is the *vanishing point* of *all perpendiculars.* A *horizontal line through s′* and *lying in the picture plane* is the *perspective of the horizon* mentioned above; *this horizontal line through s′* is known in perspective as the **horizon.** *All horizontal lines vanish in points on the horizon,* Fig. 5.

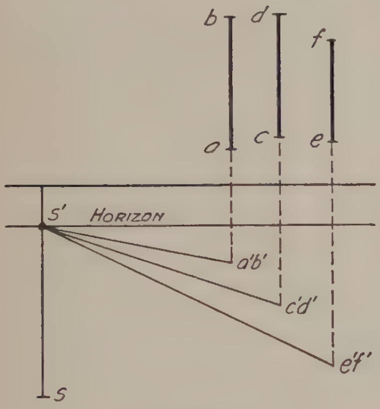


FIG. 5.

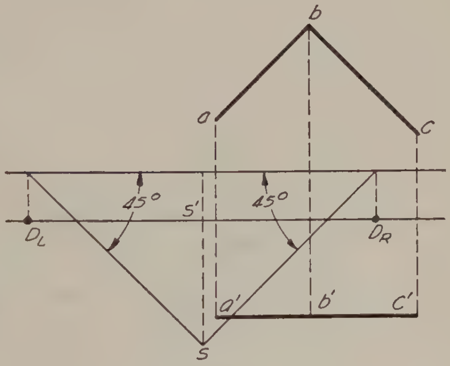


FIG. 6.

*Any horizontal line making 45 degrees with the picture plane* is known as a **diagonal,** e.g., $AB$ and $BC$, Fig. 6. *Diagonals vanish in points of the horizon,* because they are horizontal lines. On inspection of Fig. 6 it will be seen that the vanishing points $D_R$ and $D_L$ of *diagonals* are on the *horizon* at *distances to the right and left of s′ equal to the distance that s is below GL,* or that *S* is *before the picture plane.* Hence $D_R$ and $D_L$, also known as **points of distance,** are readily located whether *s* is without the limits of the drawing or not.

**160.** Problem 46. **To find the perspective of a given point by diagonals and perpendiculars.**

**Analysis:**

The point of intersection of the perspectives of two intersecting lines is the perspective of their point of intersection.

**Solution:**

If through the given point a perpendicular and a diagonal are passed, the intersection of their perspectives is the perspective of the given point.

**Construction:**

Given: Point *O*, third quadrant, Fig. 7.

Required: Perspective of *O* by a perpendicular and a diagonal.

The *V* projection *s′* of the point of sight *S* is assumed wherever desired and the *horizon* is passed through it (*1*). *s′* is the *vanishing point* of *all perpendiculars*. $D_R$ and $D_L$, the *vanishing points* of *all right* and *left diagonals*, are located at equal distances to the right and left of *s′*, which distance is equal to the distance at which *S* is assumed before the picture plane. The diagonal through *O* pierces *V* at *v′* and its perspective is $v'D_R$ (*1*,*5*). The perpendicular through *O* pierces *V* at *o′* and its perspective is *o′s′* (*6*). $v'D_R$ and *o′s′* intersect at $o_1$ the required perspective.

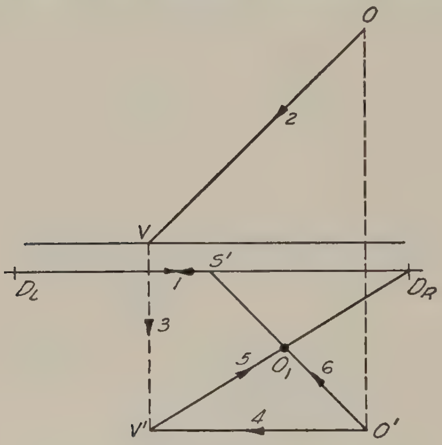


Fig. 7.

**Defect:** Though the construction explained above can be, and generally is, used in perspective construction, problems will arise in which even the points $D_R$ and $D_L$ fall without the limits of the drawing. In such cases a geometrical modification of the method is used; i.e., points known as **reduced points of distance** are used instead of $D_R$ and $D_L$. These *reduced points of distance are located* on the *horizon* at *distances to the right and left of* $s'$ *equal to some simple fractional part of* $s'D_R$, e.g., $\frac{1}{2}$, $\frac{1}{4}$, etc. Such points are lettered $D_{2R}$ and $D_{4R}$. See Fig. 8.

161. Problem 47. **To find the perspective of a given point by perpendiculars and diagonals, using reduced points of distance.**

**Construction:**

Given: Point $O$, third quadrant, Fig. 8.

Required: Perspective of $O$.

The *vanishing point* $s'$ of *verticals* is assumed as desired and the *horizon* drawn through it (*1*). The *vanishing points* of *right* and *left diagonals* are located on the $D_R$ and on the $D_L$ horizon at distances to the right and left of $s'$ equal to the distance at which $S$ is assumed before $V$ or the picture plane.

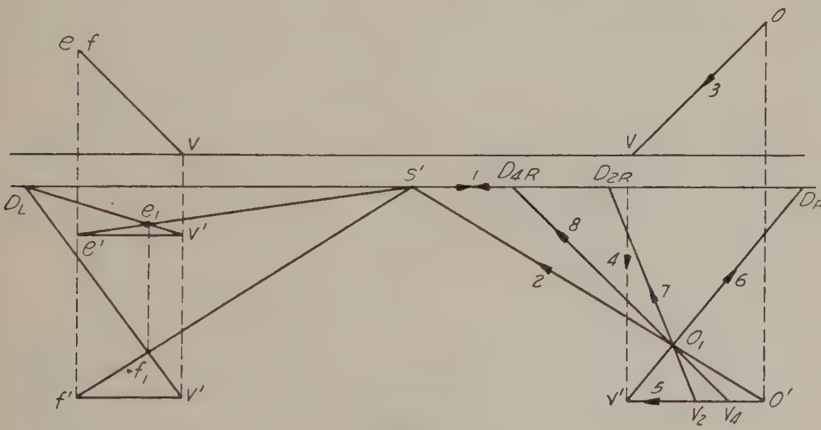


Fig. 8.

The perspective of the perpendicular through $O$ is $o's'$ (*2*). The $V$ piercing point of the diagonal is at $v'$ (*3,5*). The perspective of the diagonal is $v'D_R$ and the perspective of $O$ is $o_1$. If $D_R$ is off the limits of the drawing, then $D_{2R}$ can be used. The line from $v_2$, midway between $o'$ and $v'$, is connected with $D_{2R}$ midway between $s'$ and $D_R$ (*7*), giving $o_1$ as before; if $D_{2R}$ is likewise inaccessible, the line $v_4D_{4R}$ (*8*) may be used to locate $o_1$.

## POINTS NEAR THE LEVEL OF THE HORIZON

**162.** If, as in the case of point $E$, Fig. 8, a point is near the level of the horizon, by the method above it is not possible to get a very satisfactory point of intersection for $e_1$. Knowing that the *perspective of a vertical line is a vertical line*, we may imagine that $E$ is *dropped on a vertical line to* $F$, and the perspective of $F$ satisfactorily located at $f_1$. A vertical line through $f_1$ intersects $e's'$ in $e_1$, the required perspective.

## CIRCLES IN PERSPECTIVE

**163.** *If the plane of a given circle is parallel to the picture plane, the perspective is a true circle; its radius may be determined by finding the perspectives of the center and one point of the circumference. If the plane of the circle is oblique to the picture plane, its perspective is an ellipse.* The *major and minor axes of this ellipse can be determined;* however, for practical work this is unnecessary; a construction known as the **eight-point method** *is more easily used* and with perfect satisfaction.

**164.** Problem 48. **To find by the eight-point method the perspective of a given circle whose plane is oblique to the picture plane.**

**Solution:**

If a rhombus is circumscribed about the circle with two of its sides parallel to the picture plane ($V$) and two parallel to $H$, the perspective of this rhombus and its diagonals may be determined by methods already described. If, then, with the perspective of one of the sides of the rhombus which is parallel to $V$, a square with its diagonals, center lines, and inscribed circle, is constructed, the required eight points of intersection of these diagonals and center lines with the circle, that is, eight points of the perspective of the given circle, are determined.

**Construction:**

Given: Circle with center $O$, Fig. 9.

Required: Perspective of circle.

The circumscribed rhombus is $ABCD$. The two sides $AD$ and $BC$, which are parallel to $V$, project on $V$ in their true length. The sides $AB$ and $DC$, which are parallel to $H$, project on $H$ in their true length. The perspective of the rhombus is

$a_1b_1c_1d_1$, found by perpendiculars and diagonals [Article 160]. Since $AD$ and $BC$ are parallel to $V$, their perspectives are parallel to their $V$ projections. Then with $a_1d_1$ as a side, the square is constructed and the diagonals, center lines, and inscribed circle drawn. Inasmuch as the vanishing point of the lines $a_1b_1$ and $c_1d_1$ comes without the limits of the drawing, it is necessary to construct a second square on $b_1c_1$ to locate four of the eight points. The remaining four points of tangency of the perspective of the circle with the perspective of the rhombus are determined by the perspectives of diagonals of the rhombus and the perspectives of the two lines through their intersection $o_1$, and parallel to the respective sides of the rhombus.

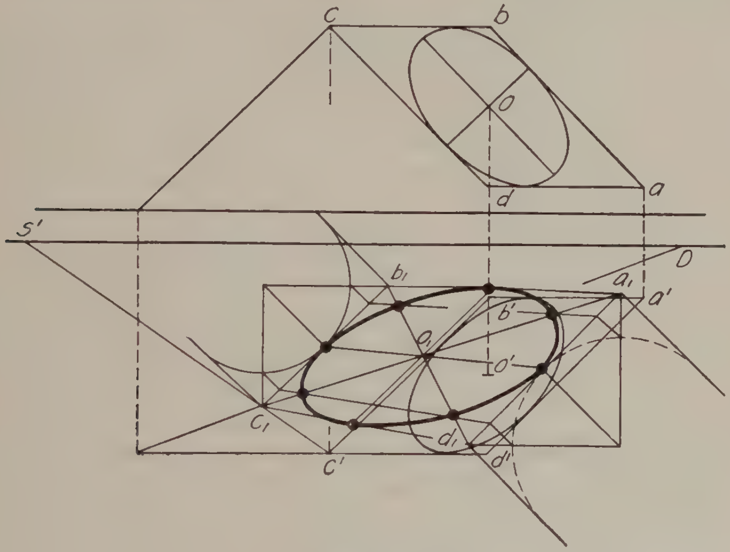


FIG. 9.

165. Problem 49. **To find the perspective of a given sphere.**

**Analysis:**

The line of tangency of a cone of visual rays tangent to the sphere and having the point of sight as an apex will be to the observer the bounding line of the body and is known as the **line of apparent contour.**

The perspective of the line of contour of any solid is the bounding line of the perspective of the solid.

**Solution:**

If with the point of sight as an apex a cone of visual rays is passed tangent to the sphere, the perspective of its line of tangency with the sphere (a circle), the apparent line of contour, is the required perspective of the given sphere.

**Construction:**

Given: Sphere *O*, Fig. 10.

Required: Perspective of the sphere.

The point of sight, *S*, is connected with *O*, the center of the sphere (*1*,*2*). *S* is revolved about a vertical line through the

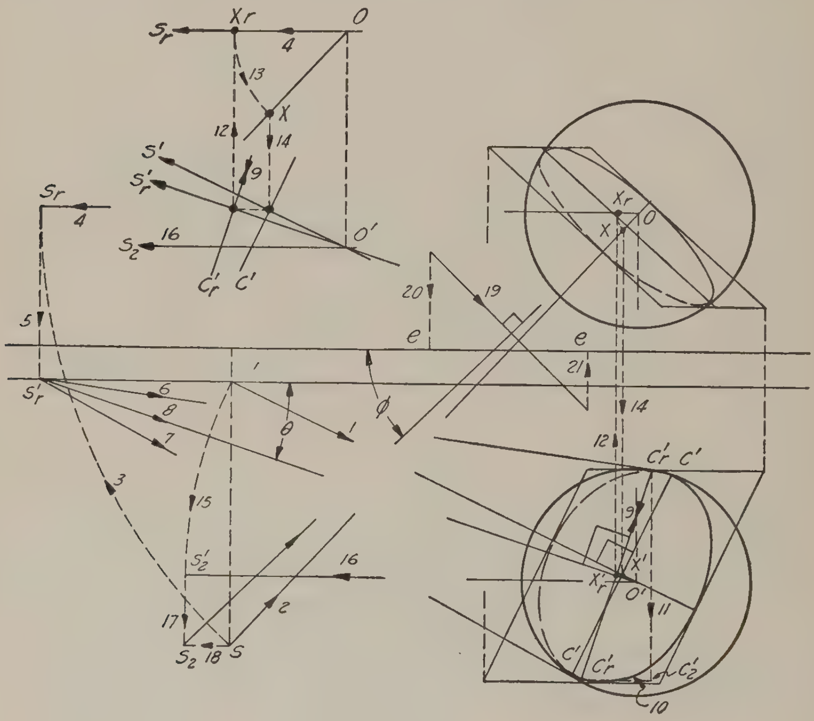


FIG. 10.

center of the sphere until *SO* is parallel to *V* (*3*,*5*). The *V* projection of $S_r$ is position $s'_r$ and the *V* projection of the revolved cone may be drawn (*6*,*9*). $c'_rc'_r$ is the *V* projection of the line of apparent contour in this revolved position and is equal to the diameter of the circular base of the cone. Angle $o's'_rs'$ equals angle "$\theta$" of the axis of the cone; hence, by projecting $c'_rc'_r$ upon a horizontal line through $c'_r$, we obtain $c'_2c'_r$, the minor axis of the elliptical $H$ projection of the circle (*10,11*). $x$ is the center of this ellipse (*12,13*). The axes of the ellipse may then be drawn through $x$ as follows: the major, perpendicular to $so$ and equal to $c'_rc'_r$, the minor, coinciding with $so$ and equal to $c'_2c'_r$. The major axis of the elliptical $V$ projection of the circular line of contour may be drawn through $x'$ perpendicular to $s'o'$ and equal to $c'_rc'_r$. To determine the minor axis, $SO$ is revolved about a perpendicular to $V$ through $O$ until it is parallel to $H$ (*14,18*). The angle between $s_2o$ and $GL$ equals angle "$\phi$" of the axis of the cone; hence a perpendicular to $s_2o$, equal in length to $c'_rc'_r$, when projected on $GL$ gives $ee$, the required minor axis (*19, 21*). The minor axis is then determined by laying off a distance equal to one half of $ee$ along $s'o'$ and on either side of $x'$.

To find the perspective of the line of contour as represented by these ellipses, the eight-point method of Problem 48 can be used. In circumscribing the projections of the rhombus we see that in both projections one pair of sides is parallel to the major axis and perpendicular to the minor.

The perspective of the sphere will be an ellipse here, which is contrary to natural observation, since it always appears circular. The reason for this apparent inconsistency is due to the fact that the perspective is found where the lines of sight pierce a *picture plane*; and, since the tangent visual cone mentioned above is a right circular cone with its axis (line $SO$) *oblique to the picture plane*, the piercing points of all the visual rays will form an elliptical perspective of the sphere.

The usual use of the picture plane is seldom strictly in agreement with nature. To agree absolutely with natural observation, we should find the perspective not on a picture plane, but on a spherical surface with its center at the point of sight. Owing to the complications encountered in finding the piercing points of the lines of sight with a sphere, a spherical perspective is seldom used.

## SHADOWS IN PERSPECTIVE

**166.** Before we take up any problems in finding the perspectives of shadows, it may be well to mention a few facts of the principles of shadows:

1. The *shadow of a given point on a given surface is the piercing point in that surface of a shadow ray through the given point.*

2. The *shadow of a given point on a given plane* is also the *intersection of the shadow ray with the projection of the shadow ray on the given plane.*

3. The *vanishing point of rays of light may be determined exactly as the vanishing point of any other group of parallel lines.* See *V*, Fig. 11.

4. *Since it is permissible to assume the direction of light, it is also permissible to assume at will the vanishing point of rays.*

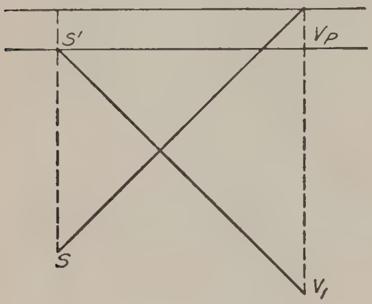


Fig. 11.

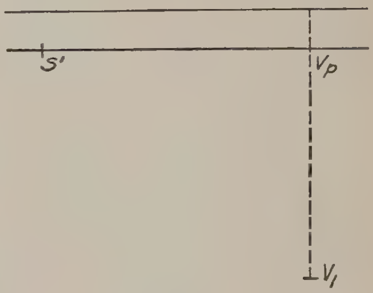


Fig. 12.

This has the added advantage of placing the shadow more nearly where it is wanted. See Fig. 12.

5. Shadows are most frequently desired on horizontal planes. The projections of rays on horizontal planes are horizontal lines, hence have their vanishing point in the horizon. *Since the ray and its projection lie in the same vertical plane, the vanishing points of rays and projections are on the same vertical line*; hence on the location of the vanishing point of rays, the vanishing point of projections on horizontal planes is immediately determined as the intersection with the horizon of a perpendicular to *GL* from $v_1$. See Fig. 12.

167. Problem 50. **To find the perspective of the shadow of a given point on a given horizontal plane.**

**Solution:**

The intersection of the perspective of the shadow ray from the given point and the perspective of the projection of this shadow ray on the given horizontal plane is the required perspective of the shadow of the given point.

**Construction:**

Given: Point $O$ and plane $T$, Fig. 13.

Required: Perspective of shadow of $O$ on $T$.

The vanishing point of rays of light is assumed at $v_1$. The vanishing point of horizontal projections is therefore located at $v_p$ on the horizon (*1*). The perspective of $O$ is determined by means of a perpendicular and a diagonal [Article 160] to be $o_1$, (*2*,*6*); $o_1v_1$ is then the perspective of the shadow ray from $O$ (*7*). The projection of $O$ on $T$ is $O_t$. Its $V$ projection is $o_t'$ (*8*) and its $H$ projection coincides with $o$. The perspective of $O_t$ is $o_t$, and the perspective of the projection $o_tv_p$ (*9*,*13*). The required perspective of the shadow of $O$ on $T$ is then $o_s$, the intersection of $o_1v_1$ and $o_tv_p$ (*7* and *13*).

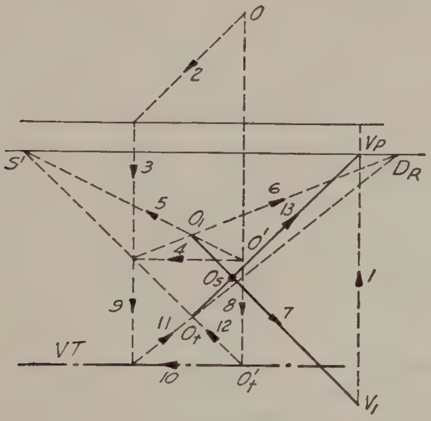


FIG. 13.

**168.** Problem 51. **To find the perspective of the shadow of a given point on any oblique plane.**

**Solution:**

The intersection of the perspective of the shadow ray from the given point with the perspective of the projection of this shadow ray on the given oblique plane is the required perspective of the shadow of the given point.

**169.** Problem 52. **To find the perspective of the shadow of a given point on a given surface.**

**Solution:**

The intersection of the perspective of the shadow ray from the given point with the perspective of line cut from the surface by a plane through the shadow ray is the required perspective of the shadow of the given point.

**Construction:**

Given: Point and cone $X$, Fig. 14.

Required: Perspective of shadow of $O$ on $X$.

The assumed projections of a ray of light through $S$ locate the vanishing point of rays of light at $v_1$. Through $O$ are passed a shadow ray, and a line to the apex $X$. These lines pierce the plane of the base of the cone in points $P$ and $P$. The plane of the two lines cuts from the plane of the base the trace $HX$ and from the surface of the cone the element $EX$. The perspective of the element $EX$ is $e_1x_1$, and the perspective of the shadow ray through $O$ is $o_1v_1$. These perspectives intersect at $o_s$, the required perspective of the shadow of $O$ on the given cone $X$.

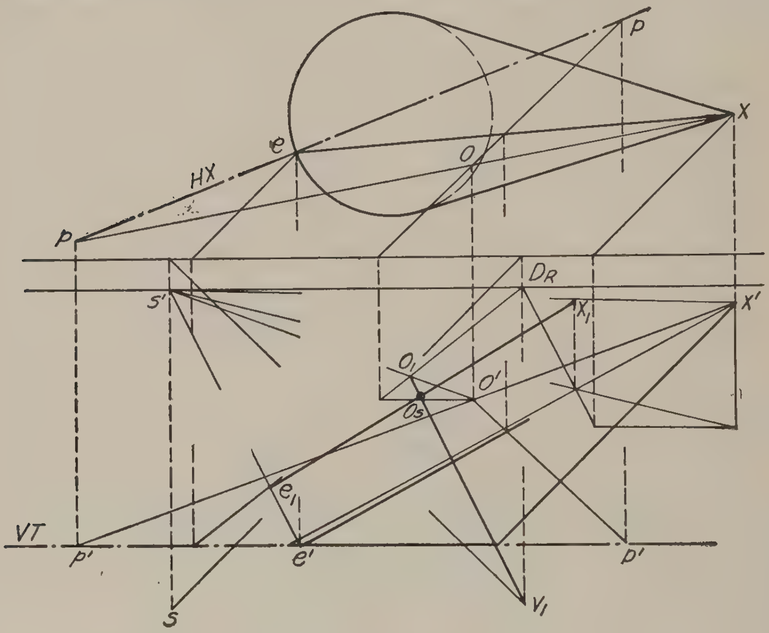


Fig. 14.

# *PART IV*

## CHAPTER XIII

## TEST EXERCISES

**170.** The following sheets of test exercises are provided for use in the classroom as well as for home study. The text presents discussions and illustrations to demonstrate the applications of the various facts of geometry to the present subject of descriptive geometry. Two varieties of exercises are required to enable the student to develop skill in using the above applications in the solution of his engineering problems. The one variety is for visual observation and mental solution only. These are given as the following "Test Exercises." The other variety involves the determination of intersections and values and must be solved graphically.

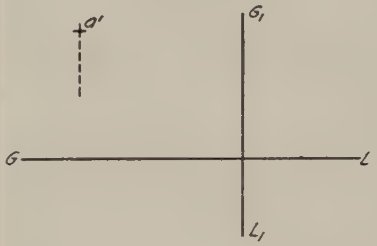
It is suggested that a brief period, five minutes is sufficient, be devoted to the appropriate exercise at each meeting of the class. Two methods may be employed. Students may be called upon for the answers to certain questions. Or cards may be ruled freehand and numbered to correspond to the arrangement of the sheet. The questions may then be answered on these cards in code, using the letter *T* for True, *F* for False, and ? if you think the given data do not justify the definite statement. The correct answer to Test 13, Problem 4, is 13–4—*aT*, *bF*, *cT*.

In each of the first three exercises only one view of the point is given; in each of the last three exercises two views of the point are given.

On a separate sheet or mentally, only, check the statements given with each problem as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

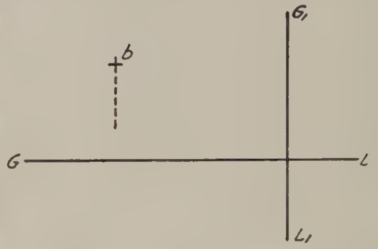
Do not mark on this sheet. Its value would thus be destroyed.

1



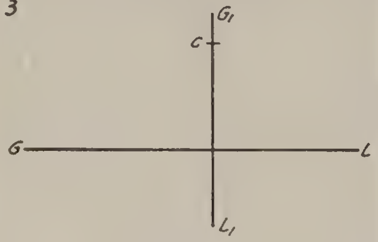
Point A is:
- a.- above the H plane
- b.- in front of V plane
- c.- in the 3rd quadrant

2



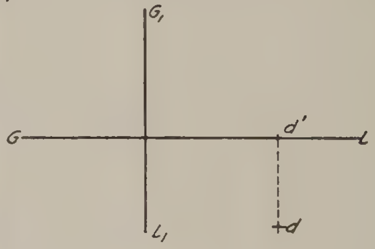
Point B is:
- a- above the H plane
- b.- in front of V plane
- c.- in the 3rd quadrant
- d.- in the H plane

3



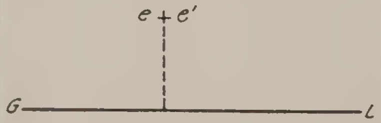
Point C is:
- a.- in the 1st quadrant
- b.- in the P plane
- c.- in the H plane
- d.- in the V plane

4



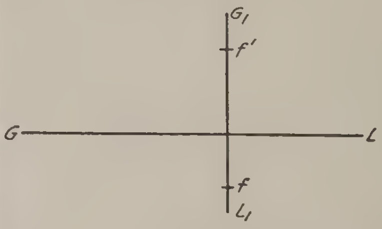
Point D is:
- a.- in the V plane
- b.- in front of V plane
- c.- below the H plane

5



Point E is:
- a.- in the 2nd quadrant
- b.- in the H plane
- c.- in front of V plane

6



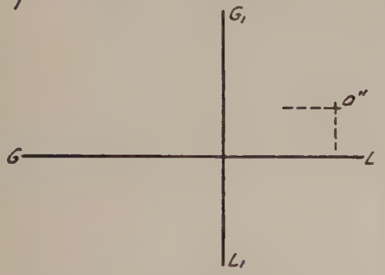
Point F is:
- a.- in the V plane
- b.- In the P plane
- c.- in the 1st quadrant

In each of the first three exercises only one view of the point is given; in each of the last three exercises two views of the point are given.

On a separate sheet or mentally, only, check the statements given with each problem as (**T**) True, (**F**) False, or (?) if the given data do not justify a definite answer.

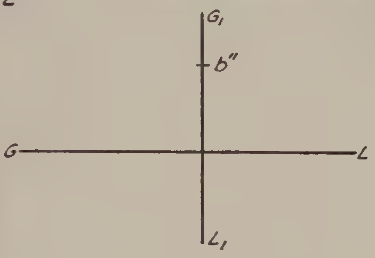
Do not mark on this sheet. Its value would thus be destroyed.

1



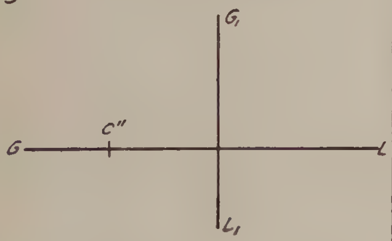
*Point A is:*
- *a.- in the V plane*
- *b.- in front of V plane*
- *c.- to right of P plane*
- *d.- in the 1st quadrant*

2



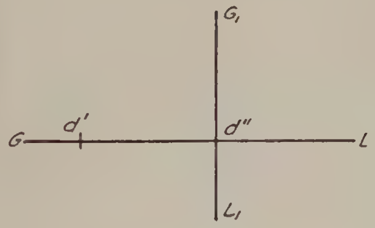
*Point B is:*
- *a.- above the H Plane.*
- *b.- in the P plane.*
- *c.- in the V plane.*

3



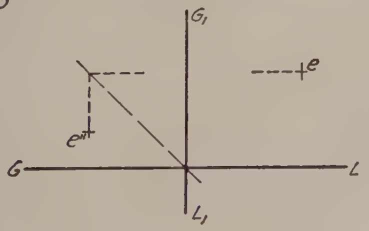
*Point C is:*
- *a.- in the V plane.*
- *b.- behind the V plane*
- *c.- in the P plane*
- *d.- to left of P plane*

4



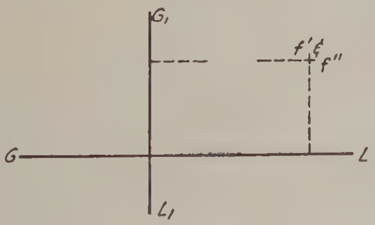
*Point D is:*
- *a.- to left of P plane*
- *b.- behind V plane*
- *c.- in the V plane*

5



*Point E is:*
- *a.- above the H plane*
- *b.- in front of V plane*
- *c.- in the 2nd quadrant*

6

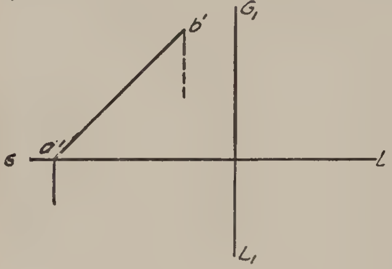


*Point F is:*
- *a.- to right of P plane*
- *b.- equidistant from H & V.*
- *c.- equidistant from V & P.*

On a separate sheet or mentally, only, check the statements given with each problem as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

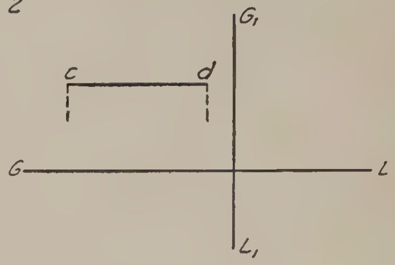
Do not mark on this sheet. Its value would thus be destroyed.

1



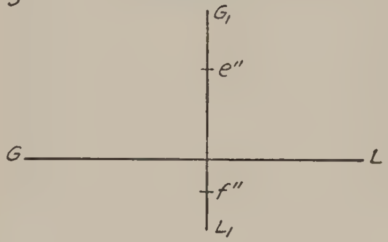
*Line AB is:*
*a.- oblique to H.*
*b.- oblique to V.*
*c.- oblique to P.*

2



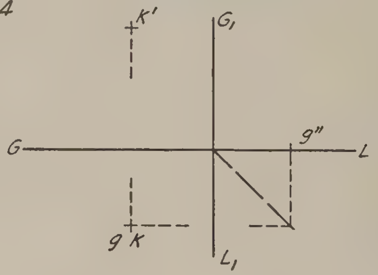
*Line CD is:*
*a- parallel to H.*
*b- above the H plane*
*c.- perpendicular to P.*

3



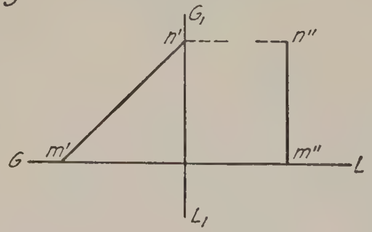
*Line EF is:*
*a.- in the P plane*
*b.- in the V plane*
*c.- perpendicular to H.*
*d.- parallel to P.*

4



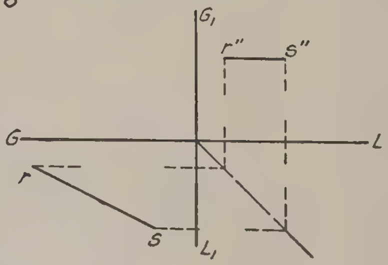
*Line GK is:*
*a.- perpendicular to V.*
*b.- parallel to V.*
*c.- perpendicular to H.*
*d.- parallel to P.*

5



*Line MN is:*
*a.- perpendicular to H.*
*b.- parallel to P.*
*c.- parallel to V.*

6



*Line RS is:*
*a.- parallel to H.*
*b.- perpendicular to V.*
*c.- perpendicular to P.*

## "IS POINT X ON LINE AB?"

On a separate sheet or mentally, only, answer the above question for each of the following problems as (T) True, (F) False, or (?) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

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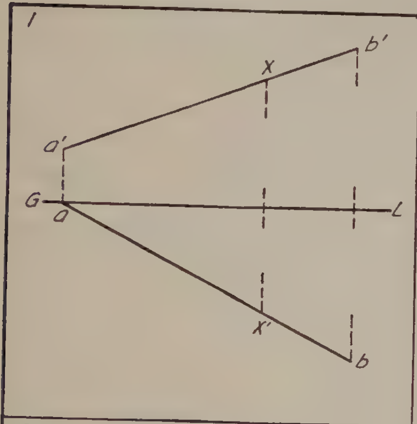

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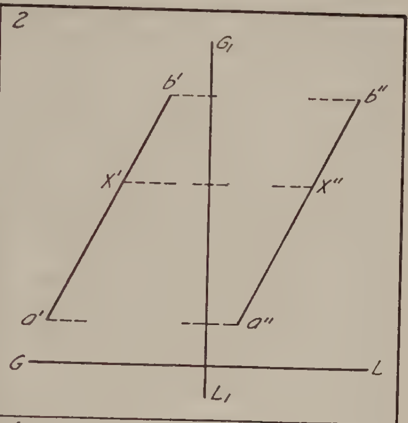

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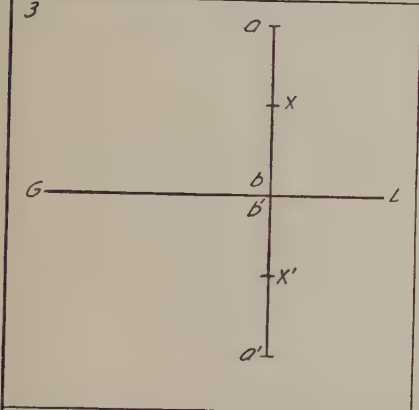

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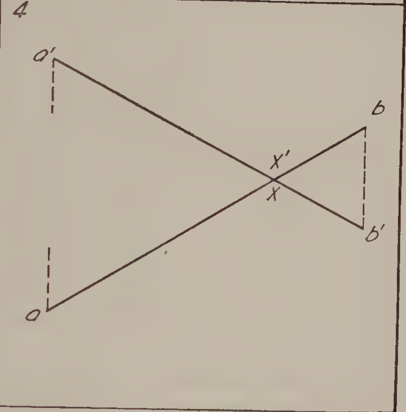

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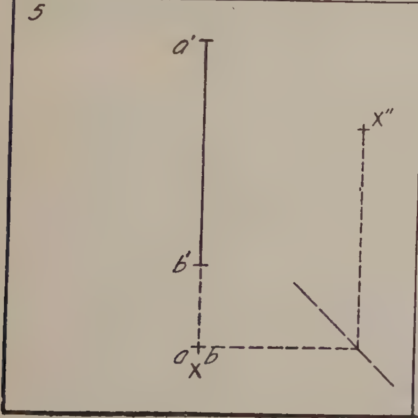

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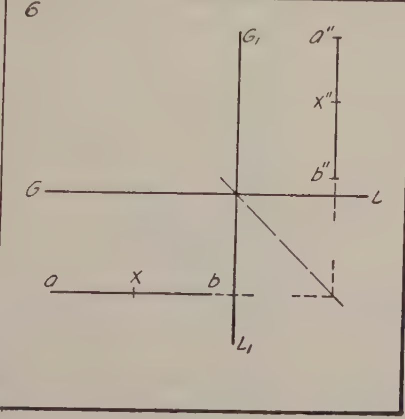
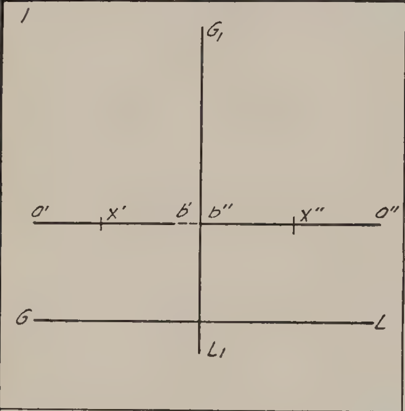
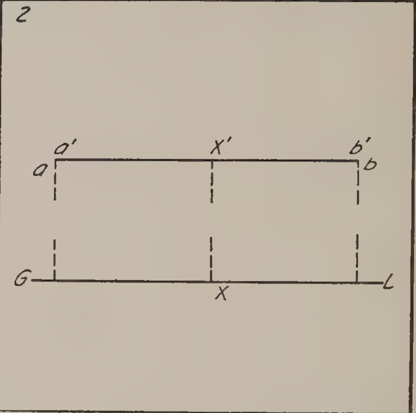


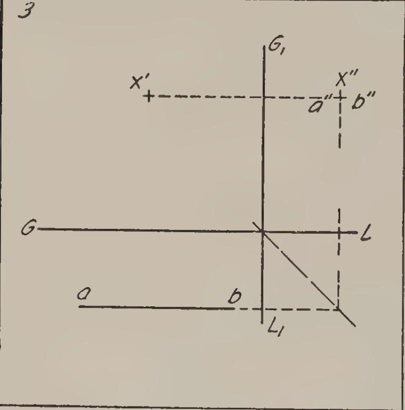
## "IS POINT *X* ON LINE *AB*?"

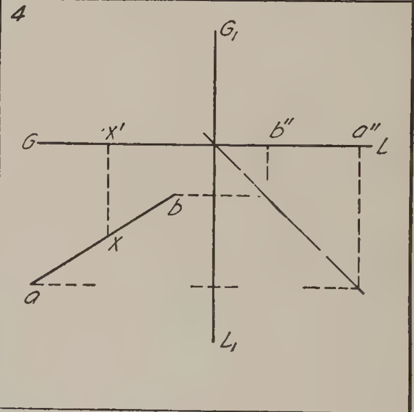
On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

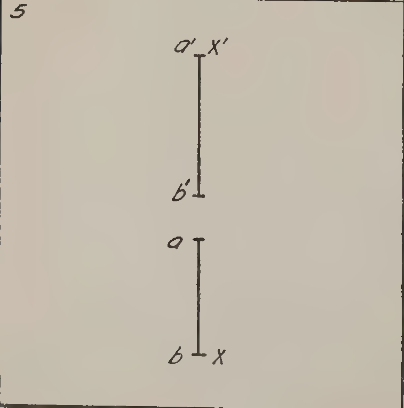
Do not mark on this sheet. Its value would thus be destroyed.

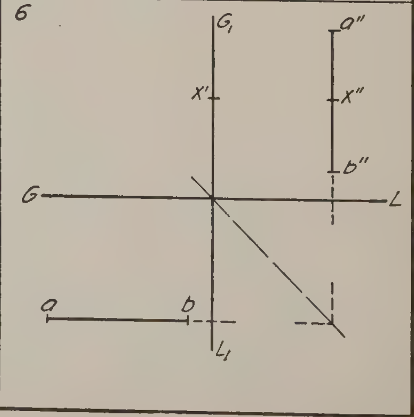








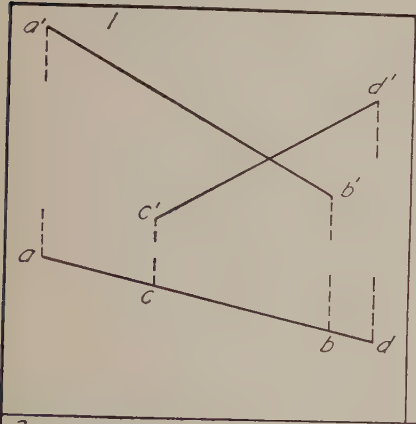


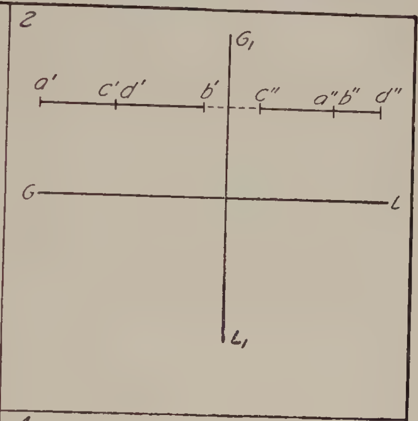


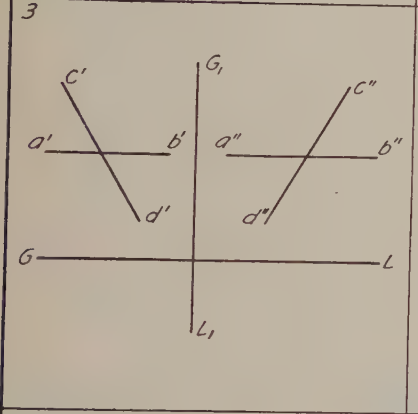
## "ARE *AB* AND *CD* INTERSECTING LINES?"

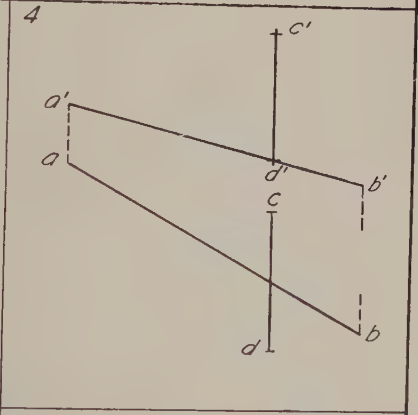
On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

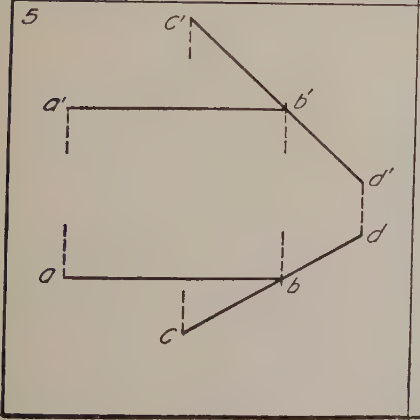
Do not mark on this sheet. Its value would thus be destroyed.

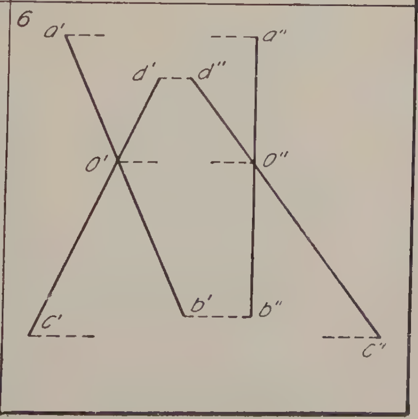










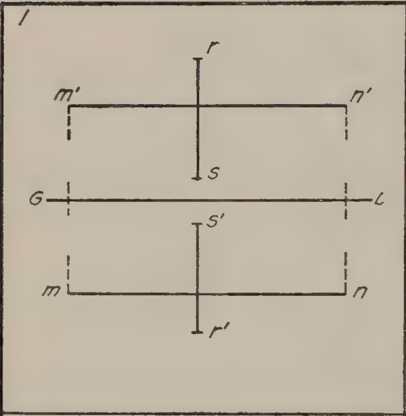


## "ARE *MN* AND *RS* INTERSECTING LINES?"

On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

1



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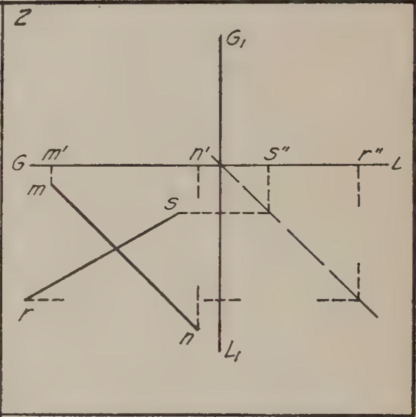

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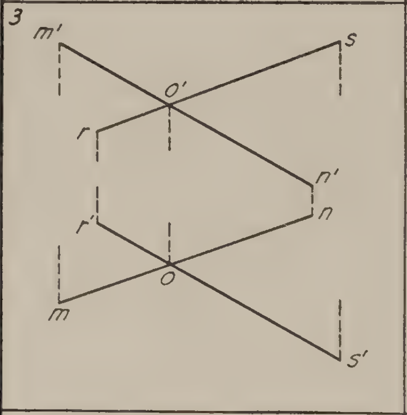

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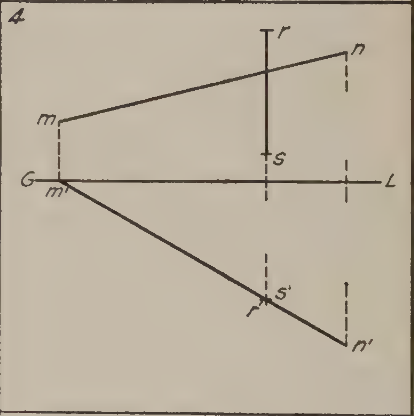

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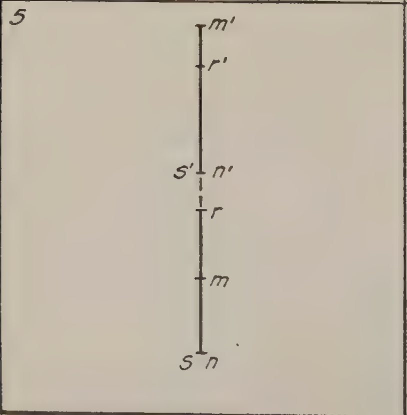

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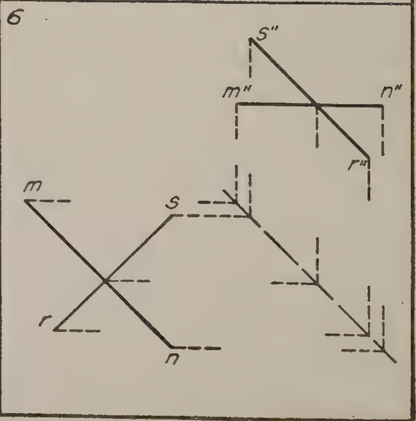
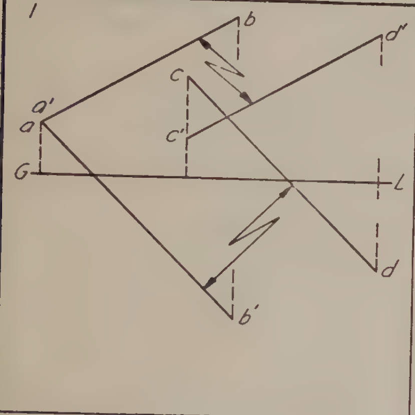


## "ARE LINES *AB* AND *CD* PARALLEL?"

On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (?) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

1



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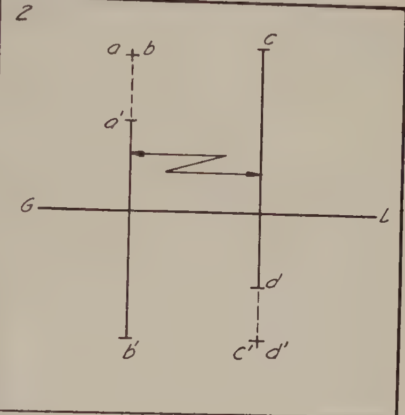

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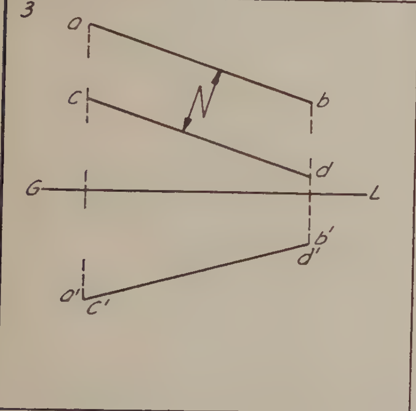

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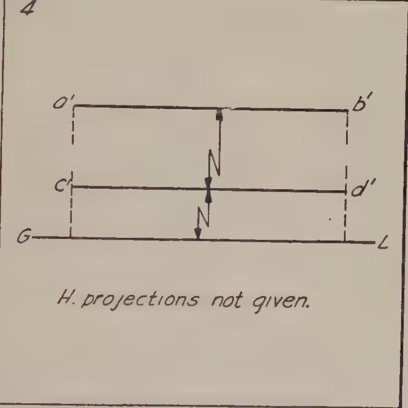
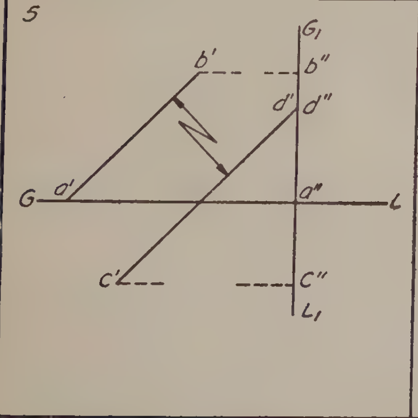


*H. projections not given.*

5



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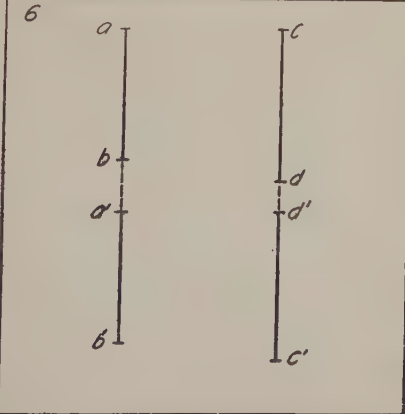
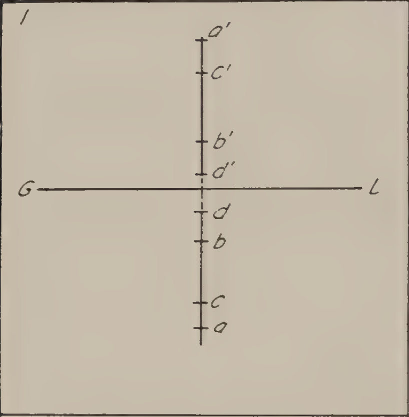


## "ARE LINES *AB* AND *CD* PARALLEL?"

On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

1



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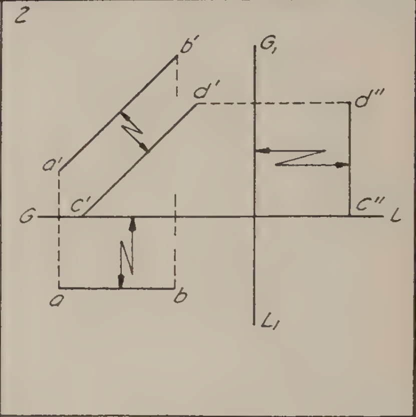

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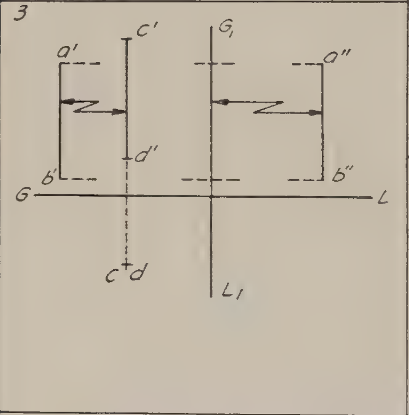

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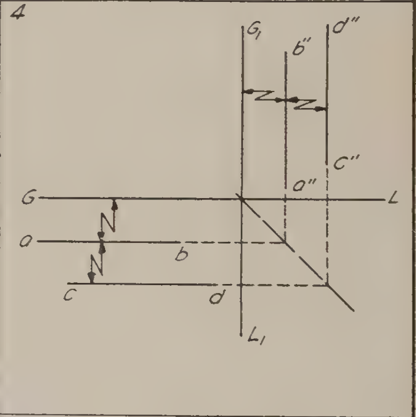

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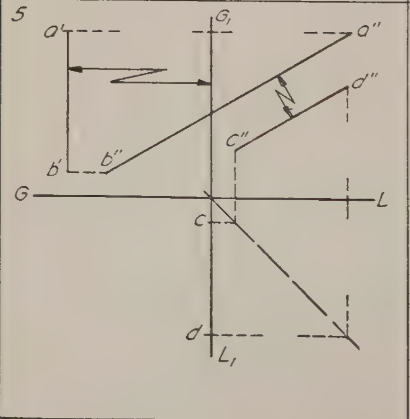

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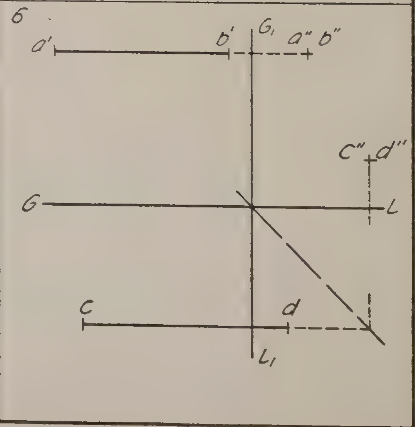


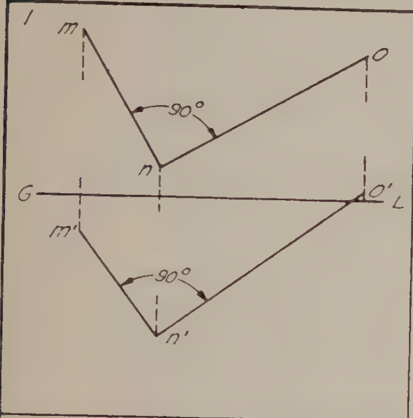
## "ARE THE GIVEN LINES PERPENDICULAR?"

(Perpendicular lines *INTERSECT* at 90°.)

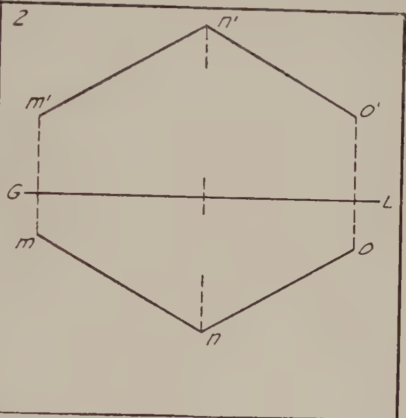
On a separate sheet or mentally, only, answer the above question for each of the following problems as **(T)** True, **(F)** False, or **(?)** if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

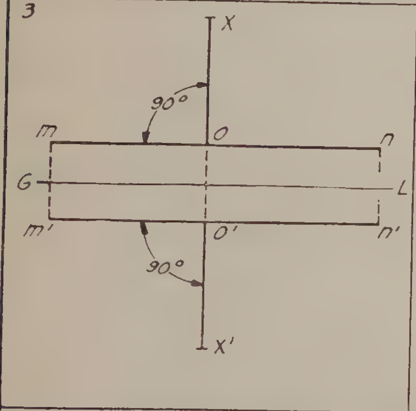
1



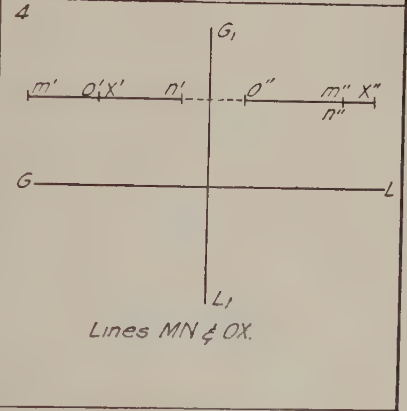
2



3

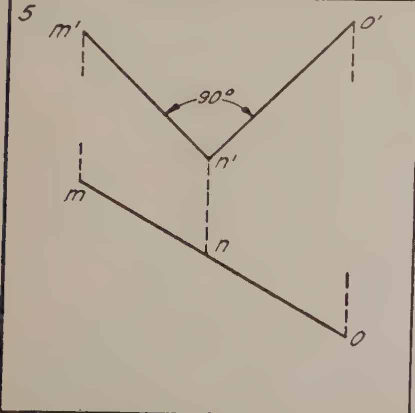


4



Lines MN & OX.

5



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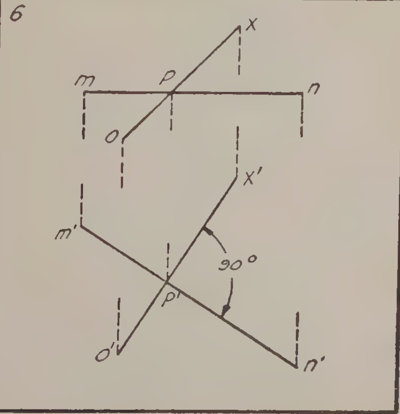
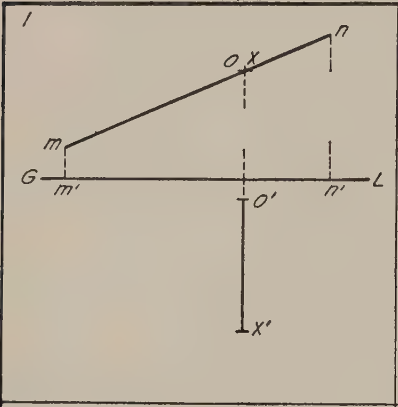


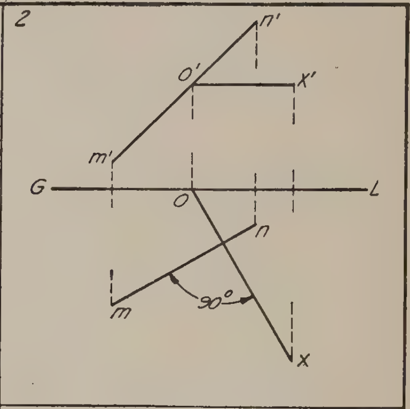
## "ARE LINES *MN* AND *OX* PERPENDICULAR?"

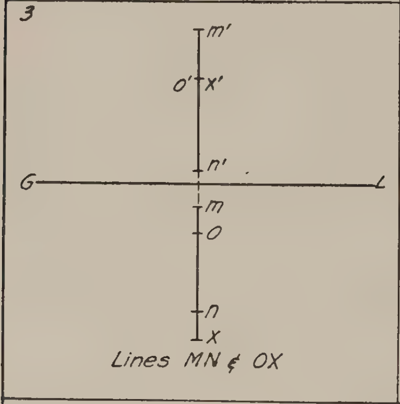
(Perpendicular lines *INTERSECT* at 90°.)

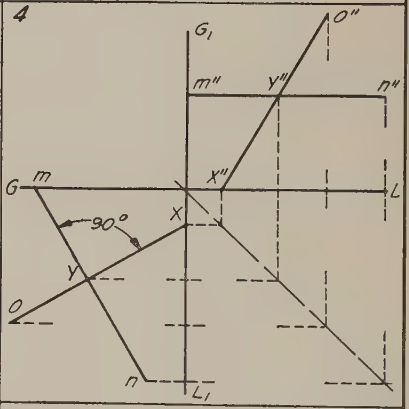
On a separate sheet or mentally, only, answer the above question for each of the following problems as **(T)** True, **(F)** False, or **(?)** if the given data do not justify a definite answer.

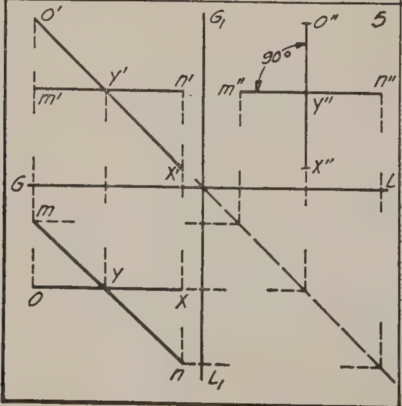
Do not mark on this sheet. Its value would thus be destroyed.

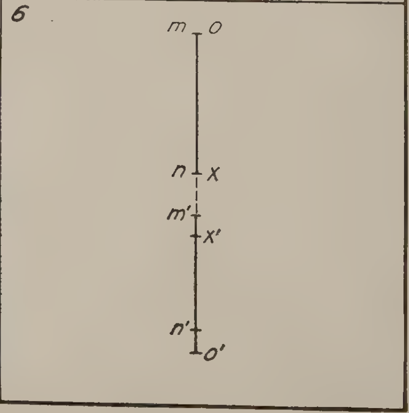








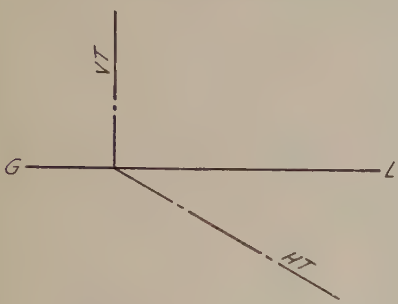




On a separate sheet or mentally, only, check the statements given with each problem as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

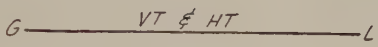
Do not mark on this sheet. Its value would thus be destroyed.

1



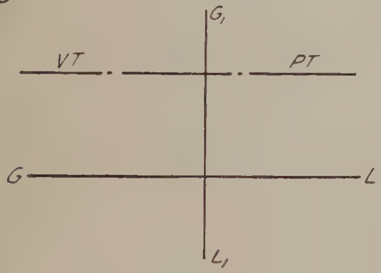
Plane T is:
- a.- oblique to P.
- b.- perpendicular to V.
- c.- parallel to H.

2



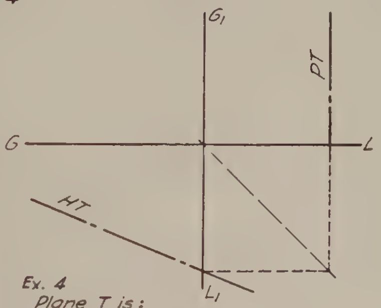
Plane T is:
- a.- oblique to H.
- b.- parallel to V.
- c.- perpendicular to P.

3



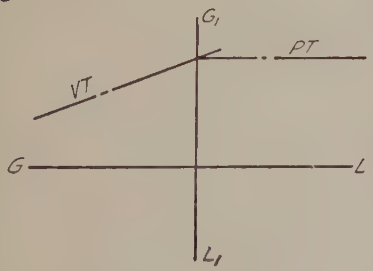
Plane T is:
- a.- oblique to H.
- b.- parallel to V.
- c.- perpendicular to P.
- d.- parallel to H.

4



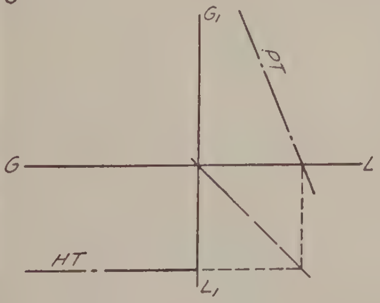
Ex. 4
Plane T is:
- a.- oblique to H.
- b.- parallel to V.
- c.- perpendicular to P.
- d.- perpendicular to H.

5



Plane T is:
- a.- parallel to H.
- b.- perpendicular to V.
- c. oblique to P.

6



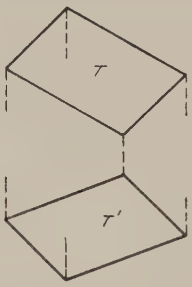
Plane T is:
- a.- oblique to H.
- b.- parallel to V.
- c.- perpendicular to P.

## TEST EXERCISE 13

On a separate sheet or mentally, only, check the statements given with each problem as (T) True, (F) False, or (?) if the given data do not justify a definite answer.

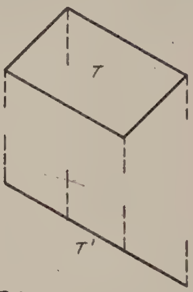
Do not mark on this sheet. Its value would thus be destroyed.

1



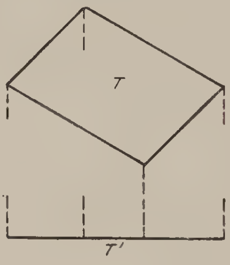
Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.
d.- perpendicular to H.

2



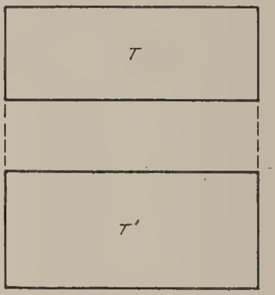
Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.
d.- perpendicular to H.

3



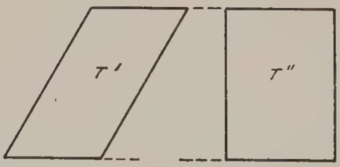
Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.

4



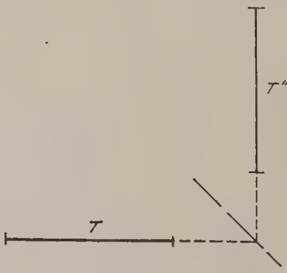
Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.

5



Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.

6



Plane T is:
a.- oblique to H.
b.- parallel to V.
c.- perpendicular to P.

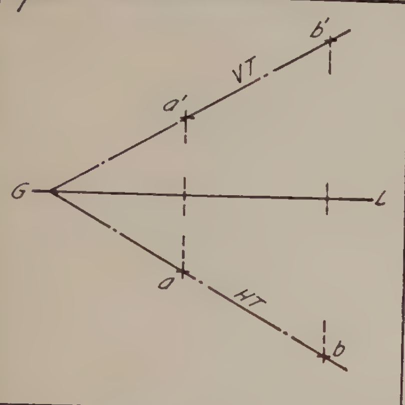
# TEST EXERCISE 14

## "DOES LINE *AB* LIE IN THE GIVEN PLANE?"

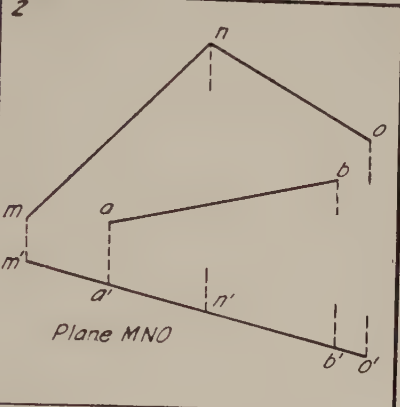
On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

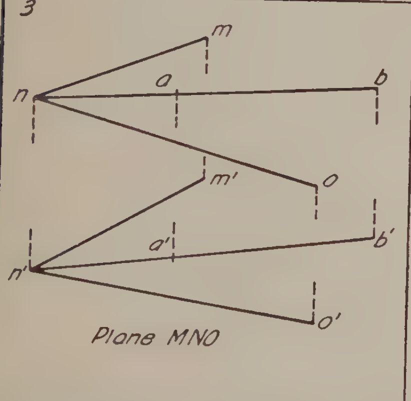
1



2



3



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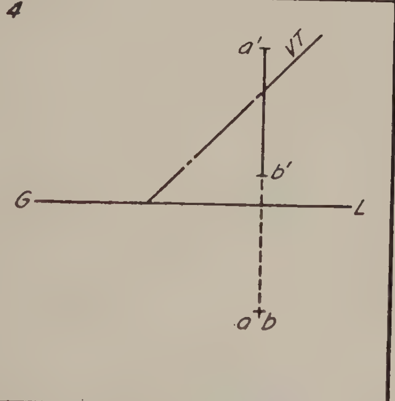
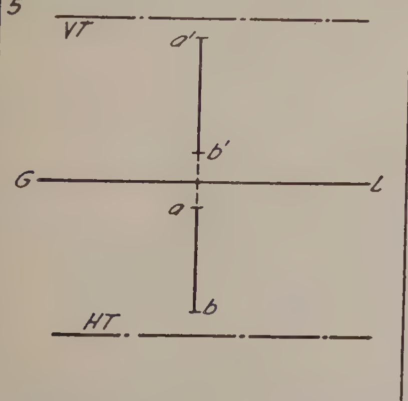
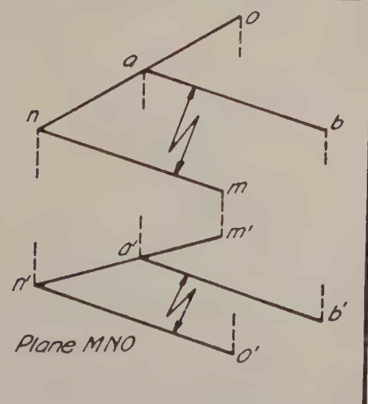


5



6

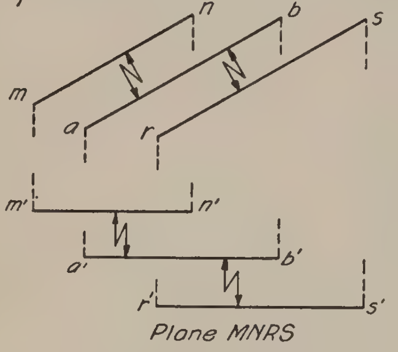


## "DOES LINE *AB* LIE IN THE GIVEN PLANE?"

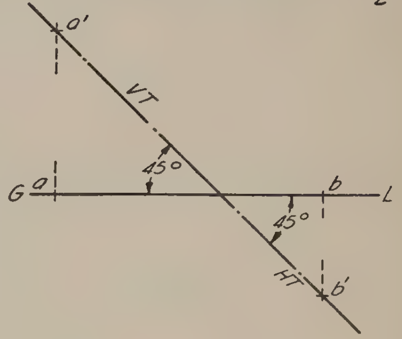
On a separate sheet or mentally, only, answer the above question for each of the following problems as (**T**) True, (**F**) False, or (**?**) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

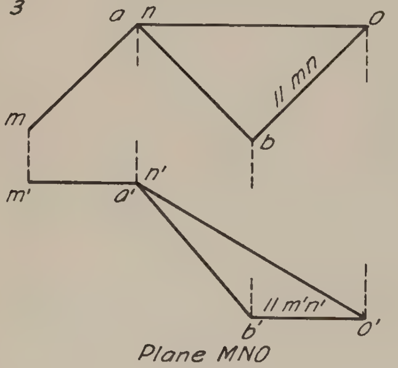
1



2



3



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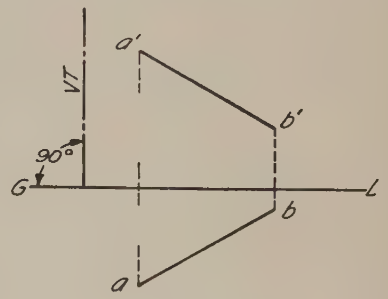
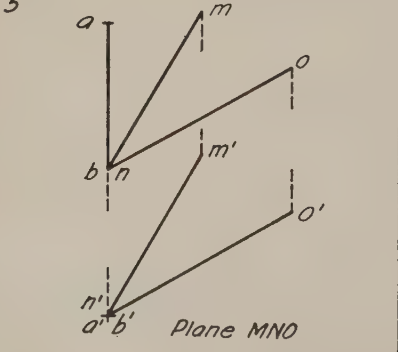
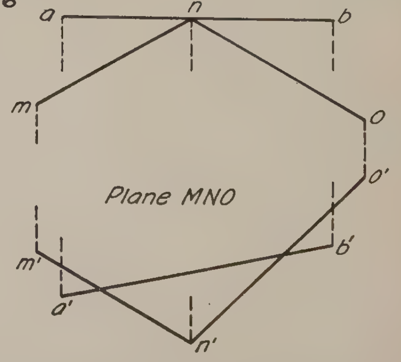


5



6

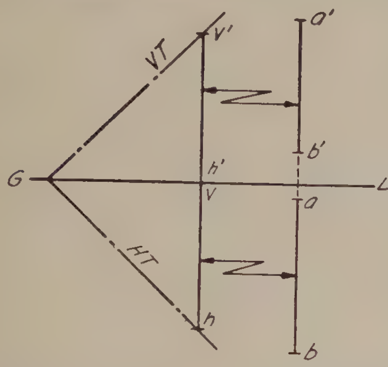


## "IS LINE *AB* PARALLEL TO THE GIVEN PLANE?"

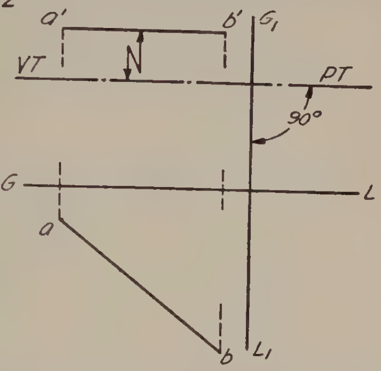
On a separate sheet or mentally, only, answer the above question for each of the following problems as (T) True, (F) False, or (?) if the given data do not justify a definite answer.

Do not mark on this sheet. Its value would thus be destroyed.

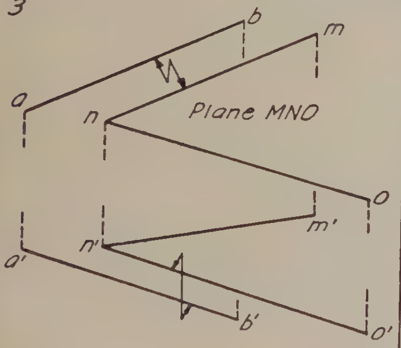
1



2



3



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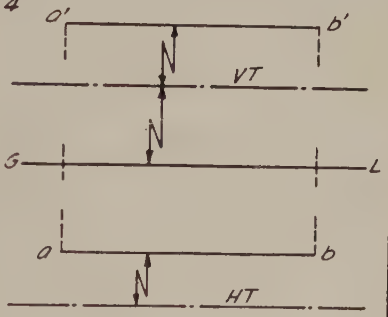

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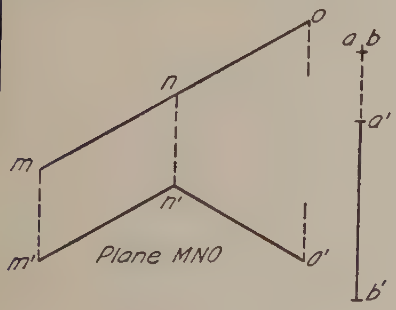

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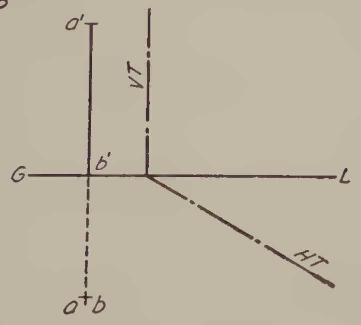

## "IS LINE *AB* PARALLEL TO THE GIVEN PLANE?"

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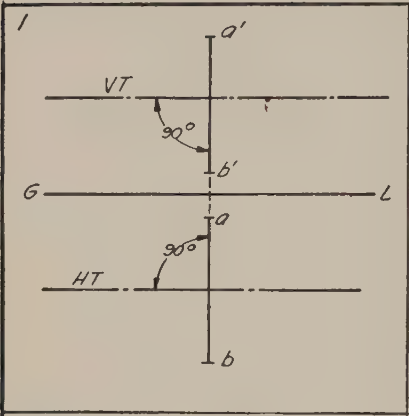

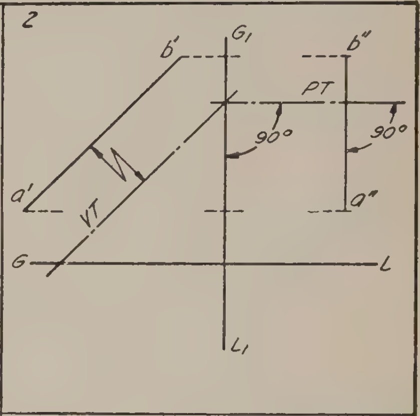

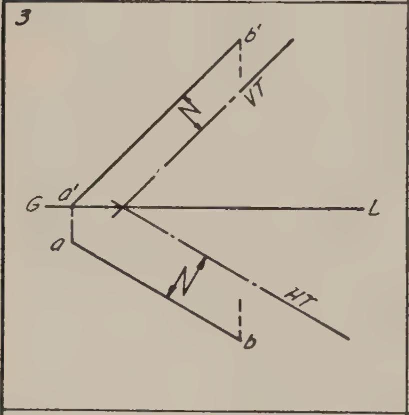

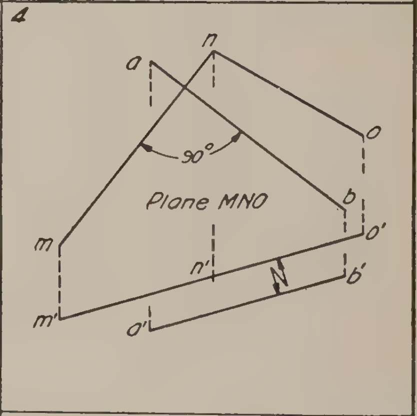

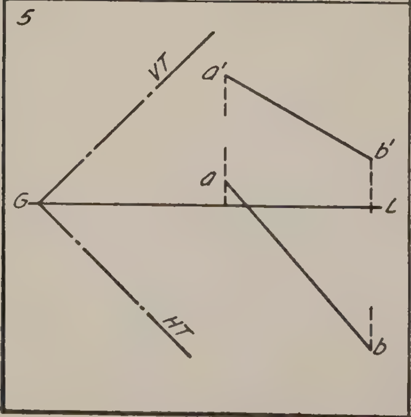

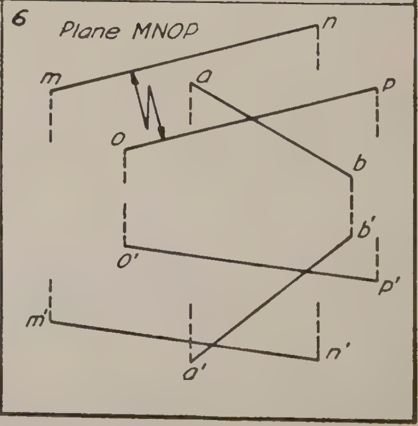

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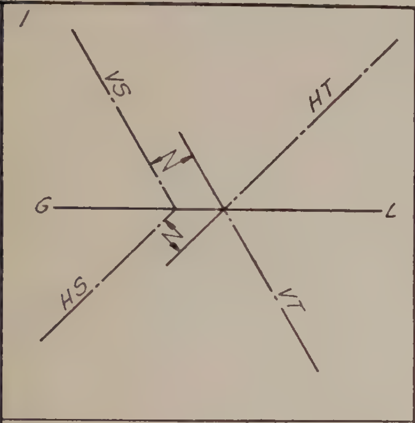

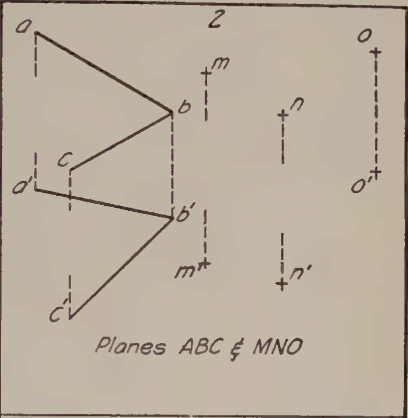

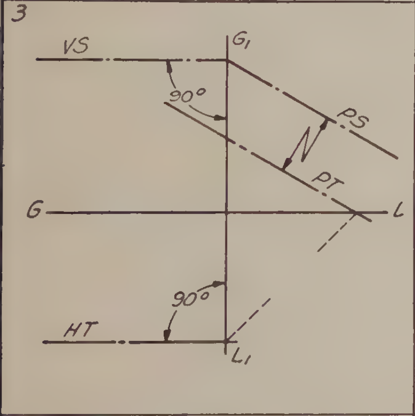

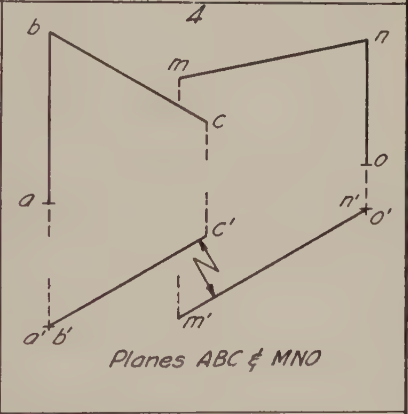

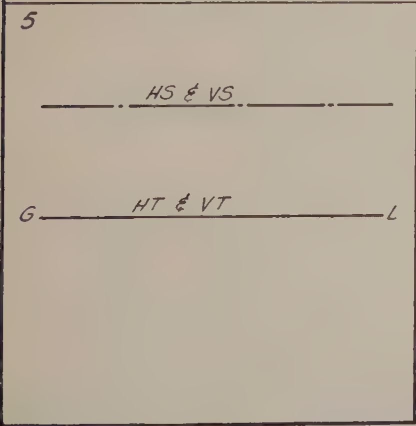

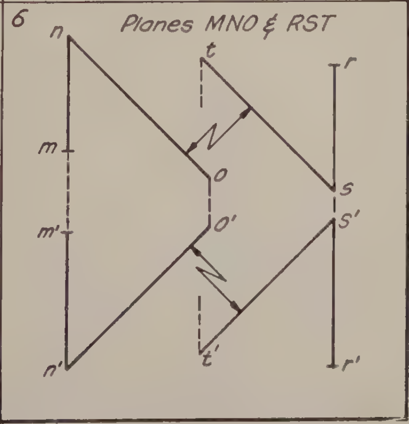

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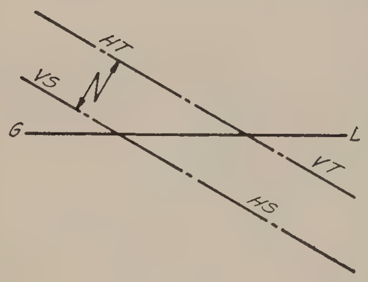

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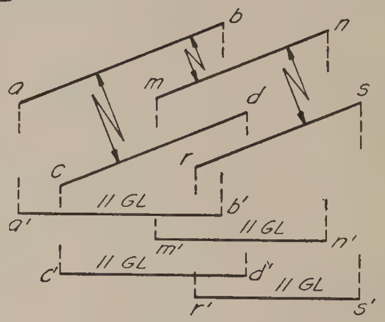
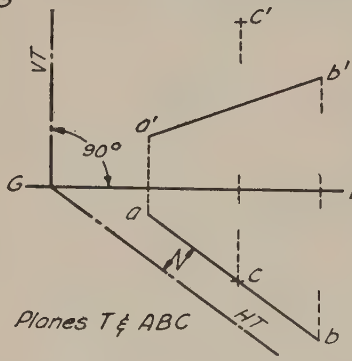


Planes ABCD & MNRS.

3



Planes T & ABC

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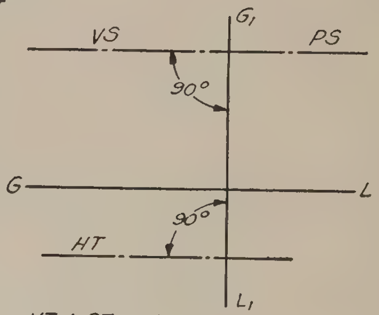

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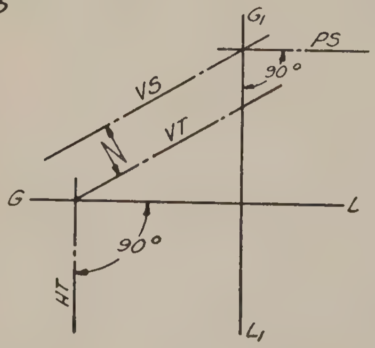

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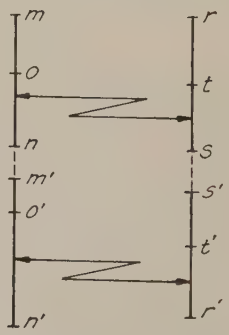

Planes MNO & RST

## "IS LINE *AB* PERPENDICULAR TO THE GIVEN PLANE?"

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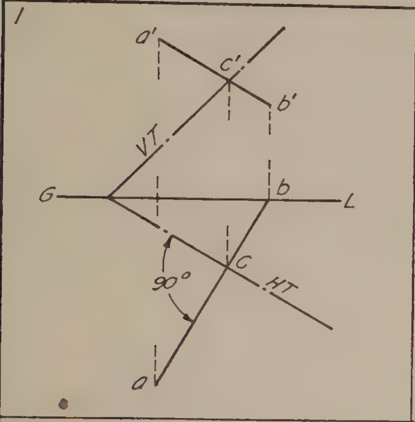

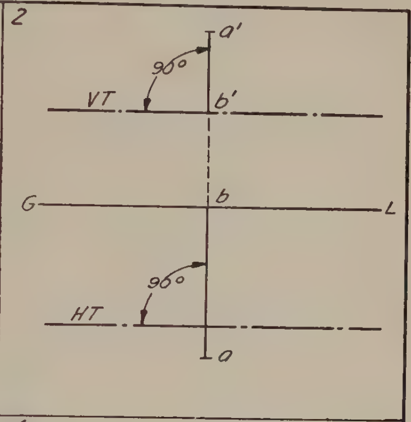

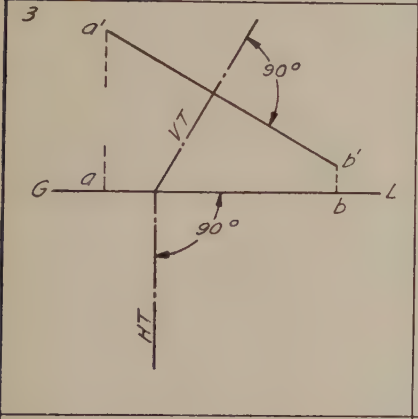

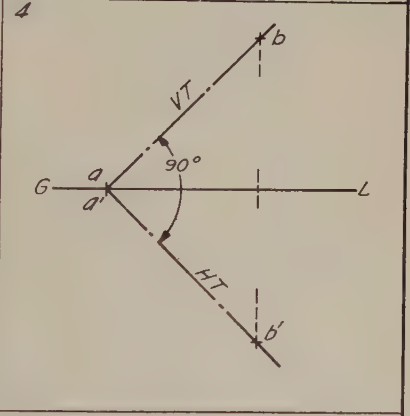

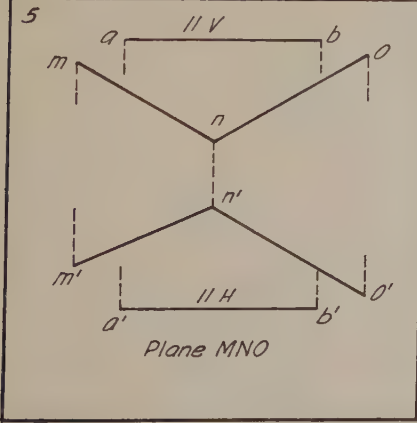

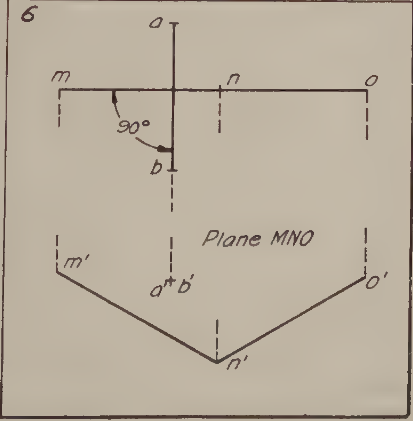

## "IS LINE *AB* PERPENDICULAR TO THE GIVEN PLANE?"

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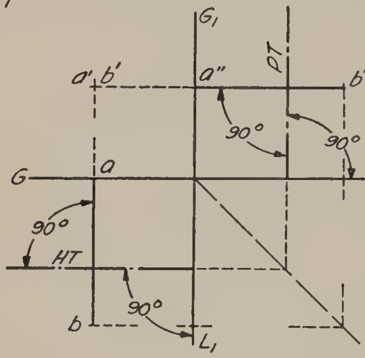

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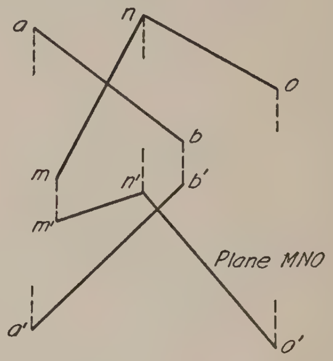

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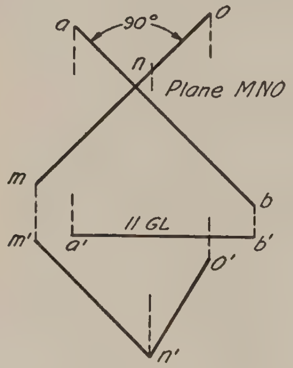

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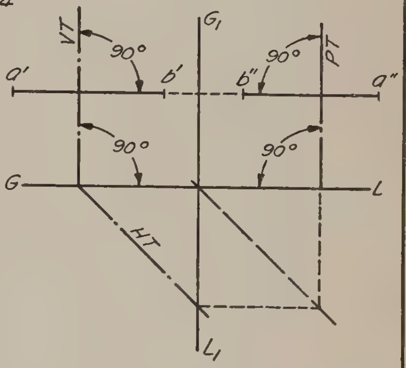

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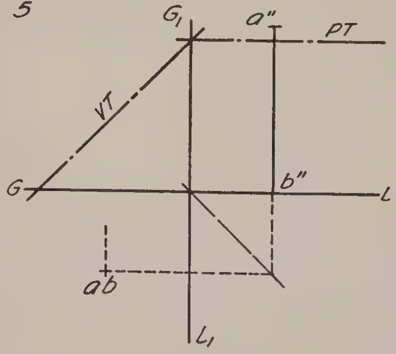

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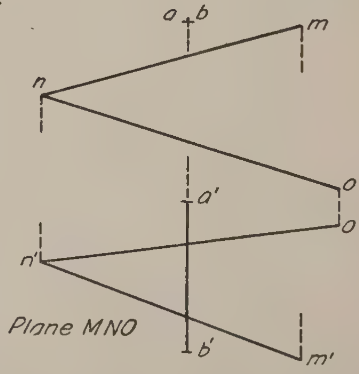

## "ARE THE GIVEN PLANES PERPENDICULAR?"

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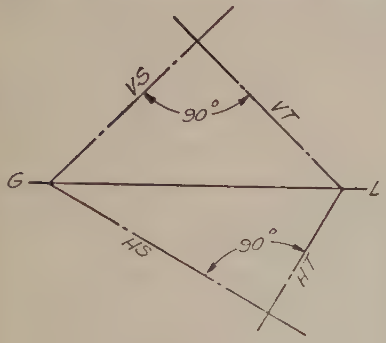

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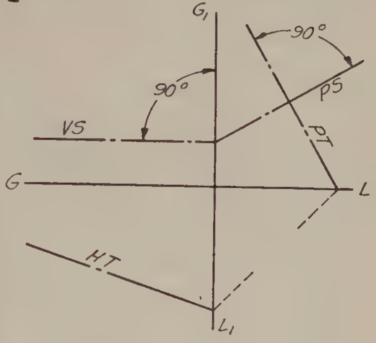

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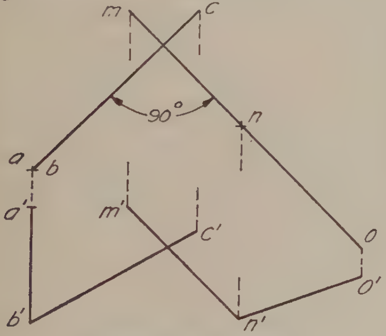

Planes ABC & MNO

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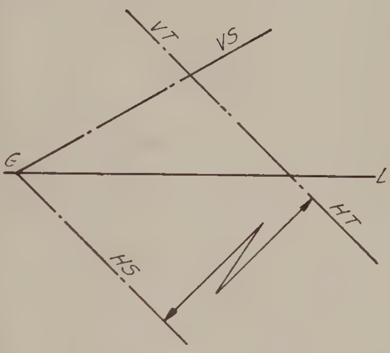

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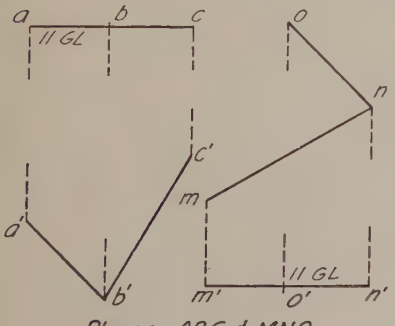

Planes ABC & MNO

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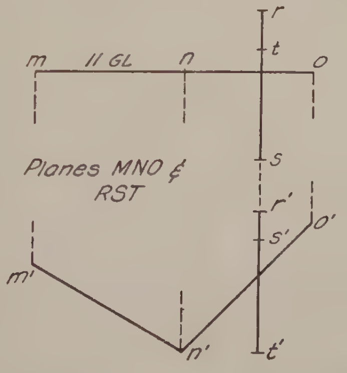

Planes MNO & RST

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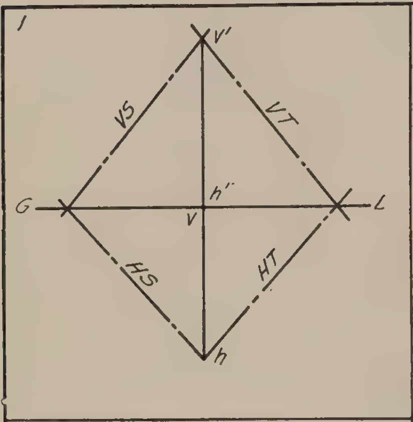

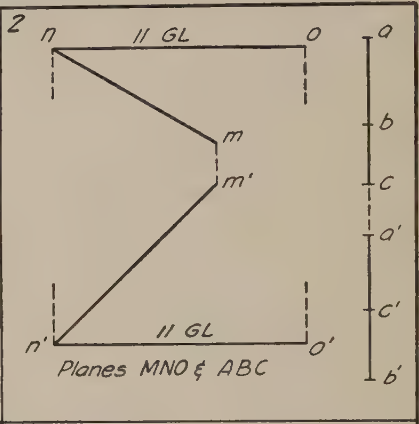

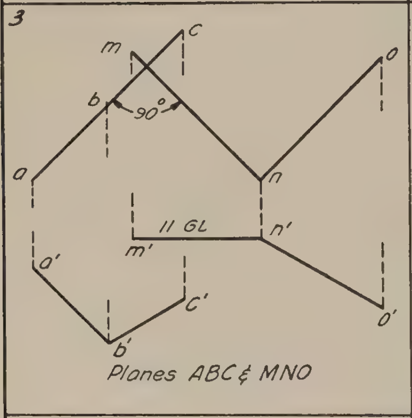

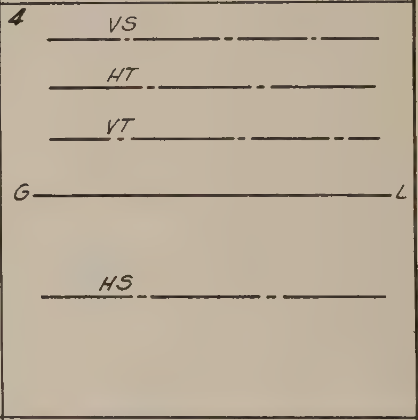

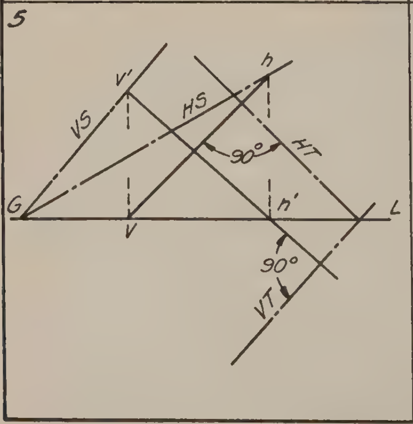

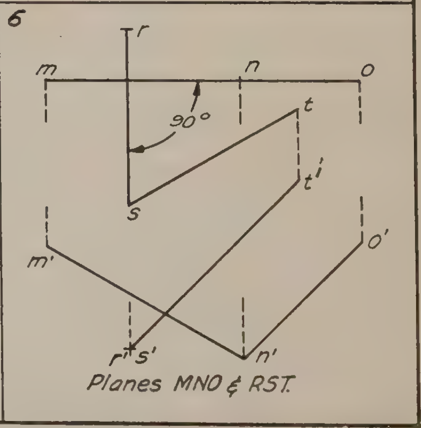

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